CSE 312

Foundations of Computing II

Lecture 5: Conditional Probability and Bayes Theorem

PAUL G. ALLEN SCHOOL
OF COMPUTER SCIENCE & ENGINEERING

Anna R. Karlin

Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

Thank you for your feedback!!!

- Several people mentioned that I was going too fast.
  - Slow me down! Ask questions!!!
  - Watch Summer 2020 videos before class (at half speed)
  - Do the reading before class.
- Some people said they wanted more practice
  - Problems in textbook
  - Do the section problems!
  - MIT "Mathematics for Computer Science" 6.042J (sections on counting & probability)
  - Get the book "A First Course in Probability" by Sheldon Ross
- More office hours?

Review Probability

Definition. A **sample space** Ω is the set of all possible outcomes of an experiment.

Examples:

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die :
$$E = \{2, 4, 6\}$$

Review Axioms of Probability

Let Ω denote the sample space and $E, F \subseteq \Omega$ be events. Note this is more general to **any** probability space (not just uniform)

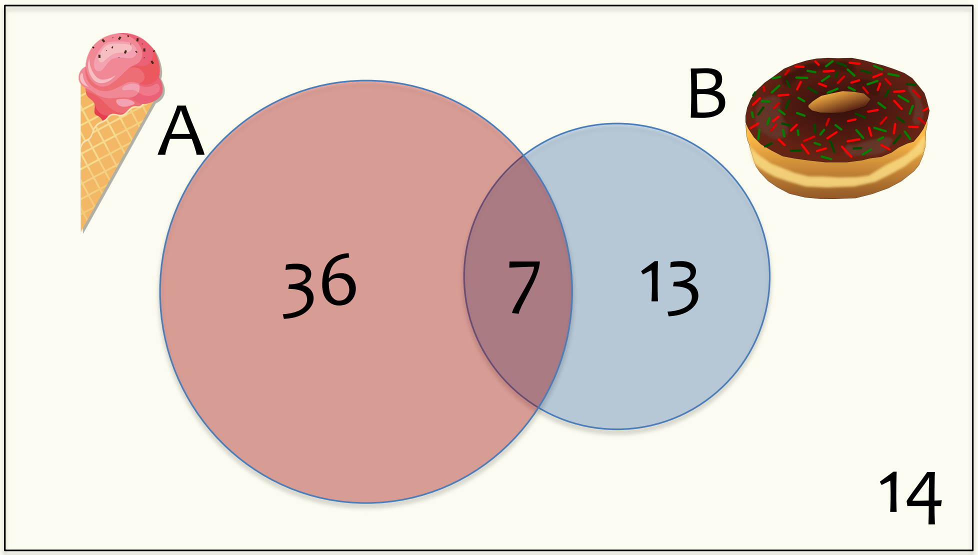
Axiom 1 (Non-negativity): $P(E) \geq 0$
Axiom 2 (Normalization): $P(\Omega) = 1$
Axiom 3 (Countable Additivity): If E and F are mutually exclusive,
then $P(E \cup F) = P(E) + P(F)$

Corollary 1 (Complementation): $P(E^c) = 1 - P(E)$
Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$
Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Agenda

- Conditional Probability
- Bayes Theorem
- Law of Total Probability
- Bayes Theorem + Law of Total Probability
- More Examples

Conditional Probability (Idea)



What's the probability that someone likes ice cream **given** they like donuts?

Conditional Probability

Definition. The **conditional probability** of event *A* **given** an event *B* happened (assuming $P(B) \neq 0$) is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

An equivalent and useful formula is

$$P(A \cap B) = P(A|B)P(B)$$

Reversing Conditional Probability

Question: Does $P(A|B) = P(B|A)$?

No! The following is purely for intuition and makes no sense in terms of probability

- Let A be the event you are wet
- Let B be the event you are swimming

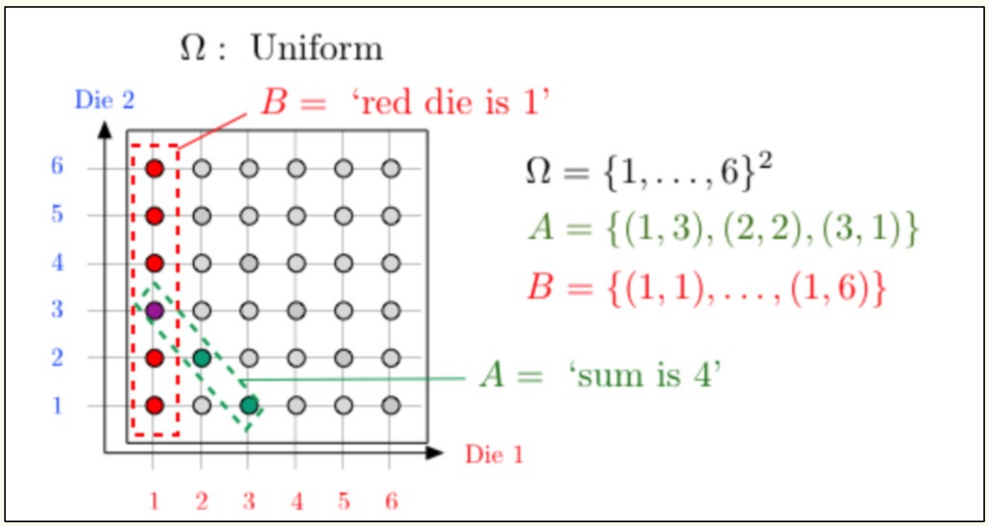
$$P(A|B) = 1$$
$$P(B|A) \neq 1$$

Example with Conditional Probability

https://pollev.com/ annakarlin185

Toss a red die and a blue die (both 6 sided and all outcomes equally likely). What is $P(B)$? What is $P(B|A)$?

	$P(B)$	$P(B|A)$
a)	1/6	1/6
b)	1/6	1/3
c)	1/6	3/36
d)	1/9	1/3



Gambler's fallacy

Assume we toss **51** fair coins.
Assume we have seen **50** coins, and they are all "tails".
What are the odds the **51^{st}** coin is "heads"?

$\mathcal{A} =$ first 50 coins are "tails"

$B =$ first 50 coins are "tails", 51^{st} coin is "heads"

$$\mathbb{P}(\mathcal{B}|\mathcal{A}) =$$

Gambler's fallacy

Assume we toss **51** fair coins.
Assume we have seen **50** coins, and they are all "tails".
What are the odds the **51**$^{\text{st}}$ coin is "heads"?

$\mathcal{A}$ = first 50 coins are "tails"

B = first 50 coins are "tails", 51$^{\text{st}}$ coin is "heads"

51$^{\text{st}}$ coin is independent of outcomes of first 50 tosses!

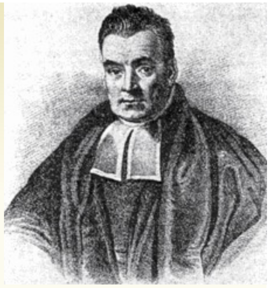
$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(\mathcal{A} \cap \mathcal{B})}{\mathbb{P}(\mathcal{A})} = \frac{1/2^{51}}{2/2^{51}} = \frac{1}{2}$$

Gambler's fallacy = Feels like it's time for " heads"!?

Agenda

- Conditional Probability
- Bayes Theorem
- Law of Total Probability
- Bayes Theorem + Law of Total Probability
- More Examples

Bayes Theorem



A formula to let us "reverse" the conditional.

Theorem. (Bayes Rule) For events A and B, where $P(A), P(B) > 0$,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$P(A)$ is called the **prior** (our belief without knowing anything)

$P(A|B)$ is called the **posterior** (our belief after learning B)

Bayes Theorem Proof

By definition of conditional probability

$$P(A \cap B) = P(A|B)P(B)$$

Swapping A, B gives

$$P(B \cap A) = P(B|A)P(A)$$

But $P(A \cap B) = P(B \cap A)$, so

$$P(A|B)P(B) = P(B|A)P(A)$$

Dividing both sides by $P(B)$ gives

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

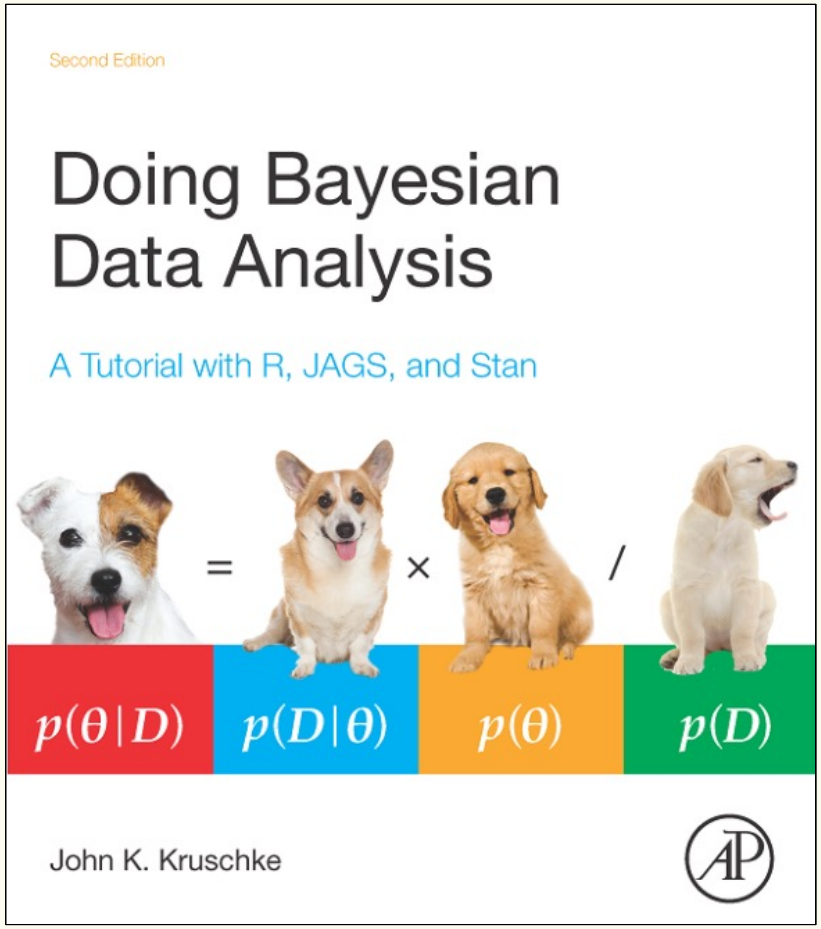
Our First Machine Learning Task: Spam Filtering

Subject: "FREE $$$ CLICK HERE"

What is the probability this email is spam, given the subject contains "FREE"?
Some useful stats:

- 10% of ham (i.e., not spam) emails contain the word "FREE" in the subject.
- 70% of spam emails contain the word "FREE" in the subject.
- 80% of emails you receive are spam.

Brain Break


Second Edition
Doing Bayesian Data Analysis
A Tutorial with R, JAGS, and Stan
= × /
p(θ|D) p(D|θ) p(θ) p(D)
John K. Kruschke

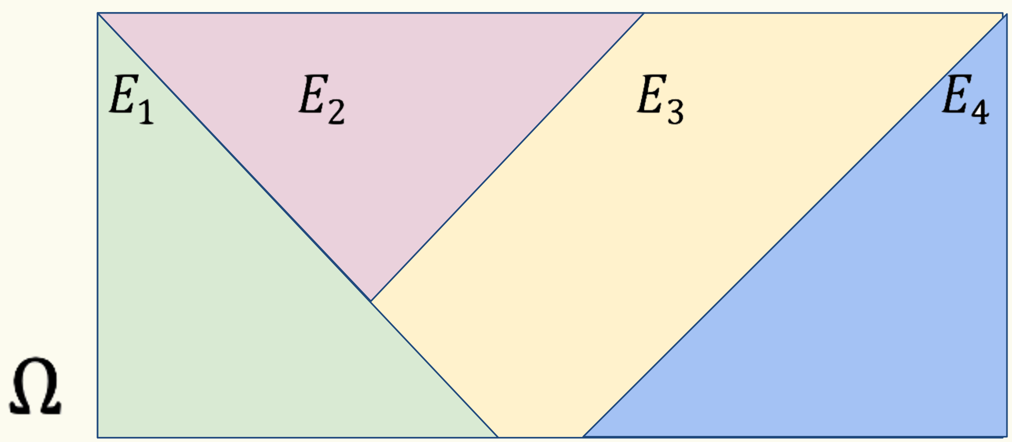
Agenda

- Conditional Probability
- Bayes Theorem
- Law of Total Probability
- Bayes Theorem + Law of Total Probability
- More Examples

Partitions (Idea)

These events **partition** the sample space

1. They "cover" the whole space
2. They don't overlap



Partition

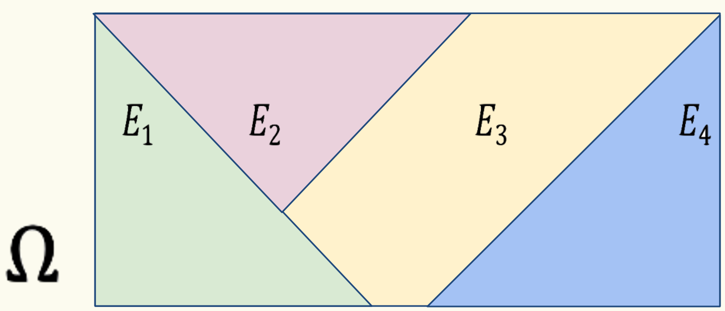
Definition. Non-empty events $E_1, E_2, \ldots, E_n$ **partition** the sample space Ω if

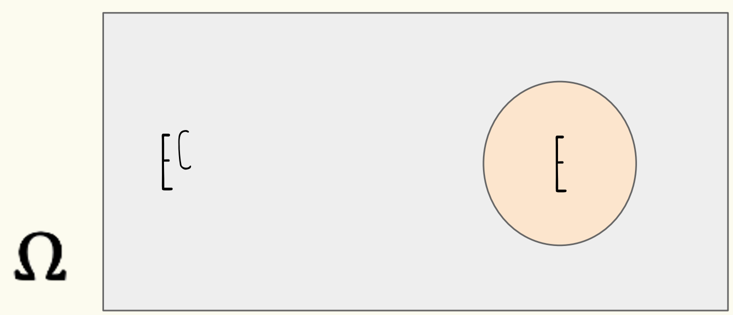
(Exhaustive)

$$E_1 \cup E_2 \cup \cdots \ \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$$

(Pairwise Mutually Exclusive)

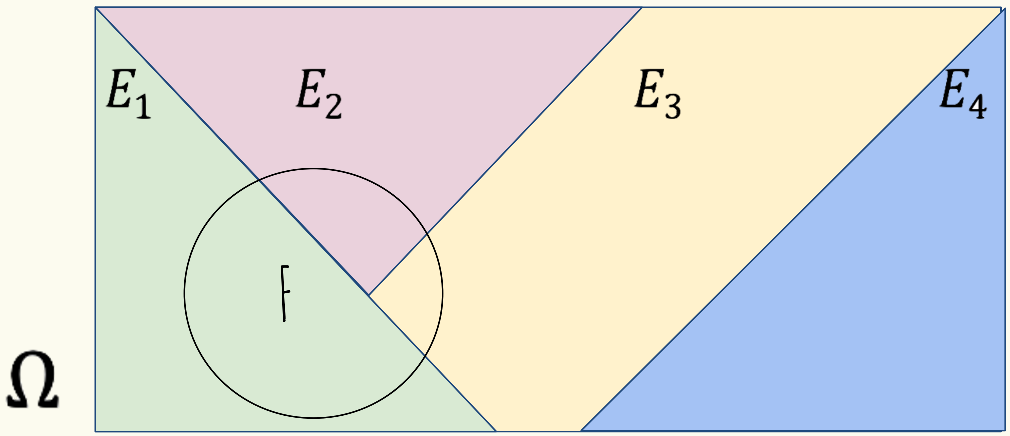
$$\forall_i \forall_{i \neq j} E_i \cap E_j = \emptyset$$





Law of Total Probability (Idea)

If we know $E_1, E_2, \ldots, E_n$ partition Ω, what can we say about $P(F)$



Law of Total Probability (LTP)

Definition. If events $E_1, E_2, \dots, E_n$ partition the sample space Ω, then for any event F

$$P(F) = P(F \cap E_1) + \ \dots + P(F \cap E_n) = \sum_{i=1}^{n} P(F \cap E_i)$$

Using the definition of conditional probability $P(F \cap E) = P(F|E)P(E)$
We can get the alternate form of this that show

$$P(F) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$

Another Contrived Example

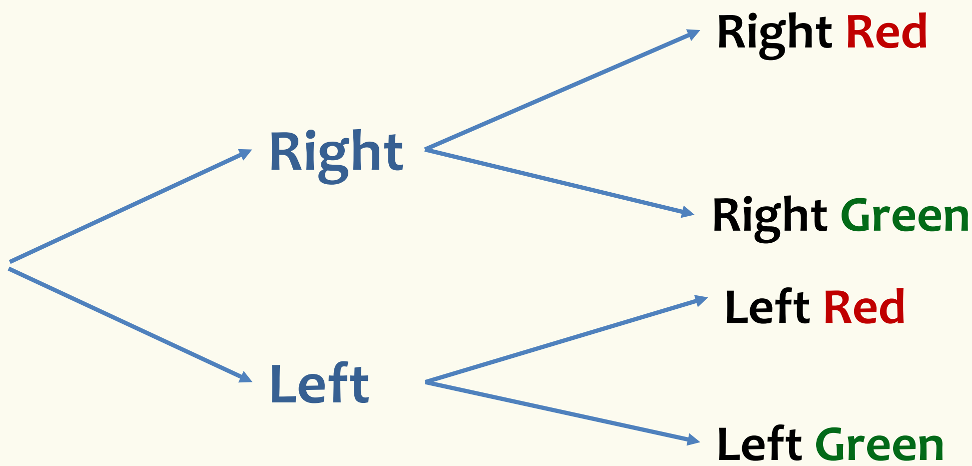
Alice has two pockets:

- **Left pocket:** Two red balls, two green balls
- **Right pocket:** One red ball, two green balls.

Alice picks a random ball from a random pocket.
[Both pockets equally likely, each ball equally likely.]

What is $\mathbb{P}(\mathbf{R})$?

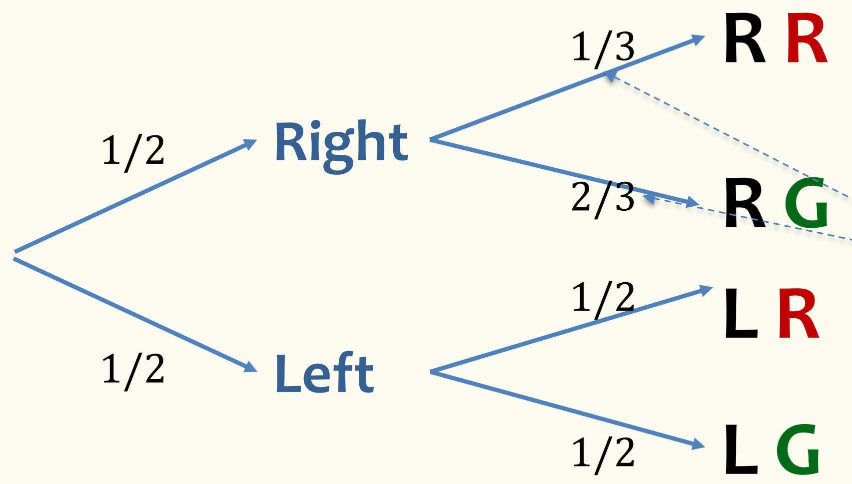
Sequential Process – Non-Uniform Case



- **Left pocket:** Two red, two green
- **Right pocket:** One red, two green.

- Alice picks a random ball from a random pocket

Sequential Process – Non-Uniform Case



- **Left pocket:** Two red, two green
- **Right pocket:** One red, two green.

1/3 = $\mathcal{P}(R \mid \mathrm{R})$ and 2/3 = $\mathcal{P}(G \mid \mathrm{R})$

$$\mathbb{P}(\mathbf{R}) = \mathbb{P}(\mathbf{R} \cap \textbf{Left}) + \mathbb{P}(\mathbf{R} \cap \textbf{Right})$$ **(Law of total probability)**

$$= \mathbb{P}(\textbf{Left}) \times \mathbb{P}(\mathbf{R} | \textbf{Left}) + \mathbb{P}(\textbf{Right}) \times \mathbb{P}(\mathbf{R} | \textbf{Right})$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

Agenda

- Conditional Probability
- Bayes Theorem
- Law of Total Probability
- Bayes Theorem + Law of Total Probability
- More Examples

Our First Machine Learning Task: Spam Filtering

Subject: “FREE $$$ CLICK HERE”

What is the probability this email is spam, given the subject contains “FREE”?
Some useful stats:

- 10% of ham (i.e., not spam) emails contain the word “FREE” in the subject.
- 70% of spam emails contain the word “FREE” in the subject.
- 80% of emails you receive are spam.

Bayes Theorem with Law of Total Probability

Bayes Theorem with LTP: Let $E_1, E_2, \dots, E_n$ be a partition of the sample space, and F and event. Then,

$$P(E_1|F) = \frac{P(F|E_1)P(E_1)}{P(F)} = \frac{P(F|E_1)P(E_1)}{\sum_{i=1}^{n} P(F|E_i)P(E_i)}$$

Simple Partition: In particular, if E is an event with non-zero probability, then

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^C)P(E^C)}$$

Agenda

- Conditional Probability
- Bayes Theorem
- Law of Total Probability
- Bayes Theorem + Law of Total Probability
- More Examples

Example – Zika Testing

Usually no or mild symptoms (rash); sometimes severe symptoms (paralysis).

During pregnancy: may cause birth defects.

Suppose you took a Zika test, and it returns "positive", what is the likelihood that you actually have the disease?

- Tests for diseases are rarely 100% accurate.

This example and following slides are from Lisa Yan (Stanford).

Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive")
- However, the test may yield a "false positive" 1% of the time
- 0.5% of the US population has Zika.

What is the probability you have Zika (event Z) if you test positive (event T).

https://pollev.com/ annakarlin185

A) Less than 0.25
B) Between 0.25 and 0.5
C) Between 0.5 and 0.75
D) Between 0.75 and 1

Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive")
- However, the test may yield a "false positive" 1% of the time
- 0.5% of the US population has Zika.

What is the probability you have Zika (event Z) if you test positive (event T).

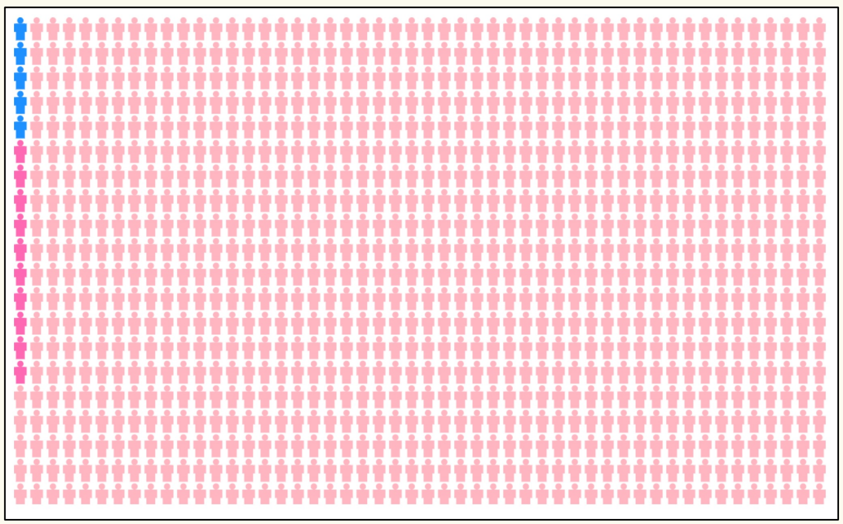
Example – Zika Testing

Have zika blue, don't pink

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive") 100%
- However, the test may yield a "false positive" 1% of the time 10/995 = approximately 1%
- 0.5% of the US population has Zika. 5% have it.

What is the probability you have Zika (event *Z*) if you test positive (event *T*).



Suppose we had 1000 people:

- 5 **have Zika** and **test positive**
- 985 **do not have Zika** and **test negative**
- 10 **do not have Zika** and **test positive**

$$\frac{5}{5+10} = \frac{1}{3} \approx 0.33$$

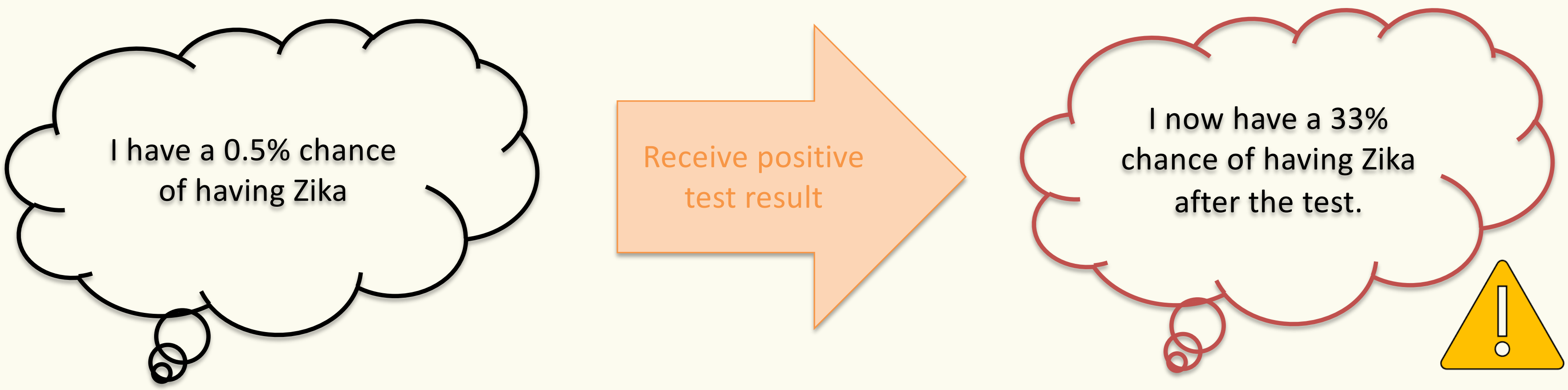
Demo

Philosophy – Updating Beliefs

While it's not 98% that you have the disease, your beliefs changed **drastically**

Z = you have Zika

T = you test positive for Zika



Prior: P(Z)

Posterior: P(Z|T)

Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive")
- However, the test may yield a "false positive" 1% of the time
- 0.5% of the US population has Zika.

What is the probability you test negative (event $\bar{T}$) if you have Zika (event Z)?

Conditional Probability Define a Probability Space

The probability conditioned on A follows the same properties as (unconditional) probability.

Example. $\mathbb{P}(\mathcal{B}^c|\mathcal{A}) = 1 - \mathbb{P}(\mathcal{B}|\mathcal{A})$

Conditional Probability Define a Probability Space

The probability conditioned on A follows the same properties as (unconditional) probability.

Example. $\mathbb{P}(\mathcal{B}^c|\mathcal{A}) = 1 - \mathbb{P}(\mathcal{B}|\mathcal{A})$

Formally. $(\Omega, \mathbb{P})$ is a probability space + $\mathbb{P}(\mathcal{A}) > 0$

➡ $(\mathcal{A}, \mathbb{P}(\cdot\,|\mathcal{A}))$ is a probability space