CSE 312

Foundations of Computing II

Lecture 7: Chain Rule and Independence

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

Announcements

- No concept check today!
- Section tomorrow is **important** with new content that you will need on pset 3, problem 7. Bring your laptops.
- I have to be out of town (and will be largely unreachable) Thursday-Saturday – Aleks will give Friday's lecture!
- Quiz 1 out later next week. Will cover material from the first two problem sets.

Friday 10/8: Bayes Theorem with Law of Total Probability

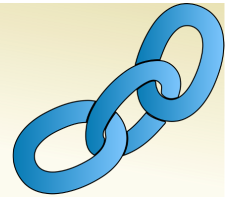
Bayes Theorem with LTP: Let $E_1, E_2, \dots, E_n$ be a partition of the sample space, and F and event. Then,

$$P(E_1|F) = \frac{P(F|E_1)P(E_1)}{P(F)} = \frac{P(F|E_1)P(E_1)}{\sum_{i=1}^{n} P(F|E_i)P(E_i)}$$

Simple Partition: In particular, if E is an event with non-zero probability, then

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^C)P(E^C)}$$

Monday 10/10: Chain Rule



$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(\mathcal{A} \cap \mathcal{B})}{\mathbb{P}(\mathcal{A})} \Rightarrow \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B}|\mathcal{A}) = \mathbb{P}(\mathcal{A} \cap \mathcal{B})$$

Theorem. (Chain Rule) For events $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$,

$$\mathbb{P}(\mathcal{A}_1 \cap \cdots \cap \mathcal{A}_n) = \mathbb{P}(\mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_2|\mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_3|\mathcal{A}_1 \cap \mathcal{A}_2)$$

$$\cdots \mathbb{P}(\mathcal{A}_n|\mathcal{A}_1 \cap \mathcal{A}_2 \cap \cdots \cap \mathcal{A}_{n-1})$$

An easy way to remember: We have n tasks and we can do them sequentially, conditioning on the outcome of previous tasks

Monday: Independence

Definition. Two events $\mathcal{A}$ and $\mathcal{B}$ are (statistically) **independent** if

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}).$$

Alternatively,

- If $\mathbb{P}(\mathcal{A}) \neq 0$, equivalent to $\mathbb{P}(\mathcal{B}|\mathcal{A}) = \mathbb{P}(B)$
- If $\mathbb{P}(\mathcal{B}) \neq 0$, equivalent to $\mathbb{P}(\mathcal{A}|\mathcal{B}) = \mathbb{P}(\mathcal{A})$

"The probability that $\mathcal{B}$ occurs after observing $\mathcal{A}$" -- Posterior
= "The probability that $\mathcal{B}$ occurs" -- Prior

Agenda

- A Sequential Process Defined Using Independence
- Independence As An Assumption
- Sometimes Independence Occurs for Nonobvious Reasons
- Conditional Independence
- Correlation vs Causation
- Information Cascades

Example – Throwing A Die Repeatedly

Alice and Bob are playing the following game.

A 6-sided die is thrown, and each time it's thrown, regardless of the history, it is equally likely to show any of the six numbers

If it shows 1, 2 → Alice wins.
If it shows 3 → Bob wins.
Otherwise, play another round

What is Pr(Alice wins on 1^{st} round) = $\frac{1}{3}$

...

Pr(Alice wins on i^{th} round) = ?

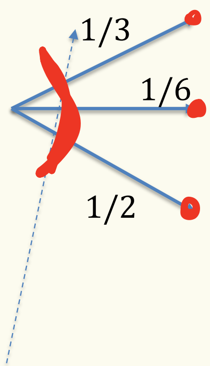
Pr(Alice wins) = ?

Sequential Process – defined in terms of independence

A 6-sided die is thrown, and each time it's thrown, regardless of the history, it is equally likely to show any of the six numbers

Local Rules: In each round

- If it shows 1,2 → Alice wins
- If it shows 3 → Bob wins
- Else, play another round



$\boldsymbol{i_{12}}$ = i th round toss 1 or 2

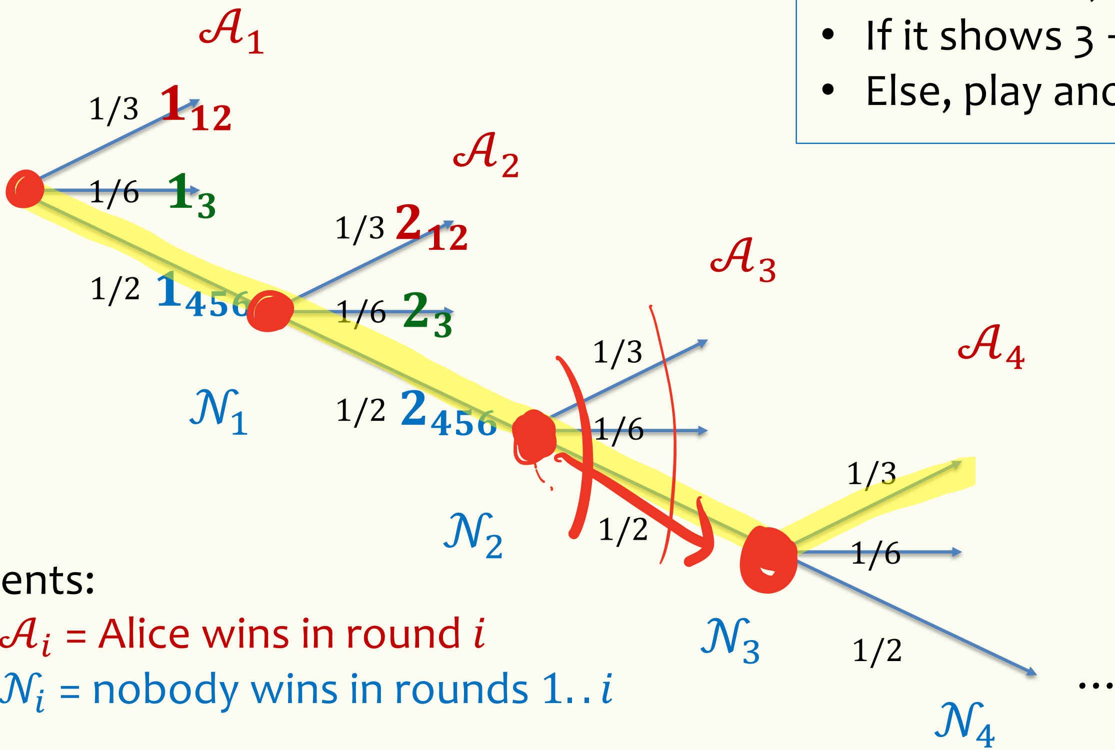
$\boldsymbol{i_{3}}$ = i th round toss 3

$\boldsymbol{i_{456}}$ = i th round toss 4 or 5 or 6

Pr (Alice wins on i -th round | nobody won in rounds $1..i$-1) = 1/3

Sequential Process – Example

Local Rules: In each round
- If it shows 1,2 → Alice wins
- If it shows 3 → Bob wins
- Else, play another round



Events:
- $\mathcal{A}_i$ = Alice wins in round i
- $\mathcal{N}_i$ = nobody wins in rounds $1..i$

$Pr(N_i \mid N_{i-1}) = \frac{1}{2}$

Sequential Process – Example

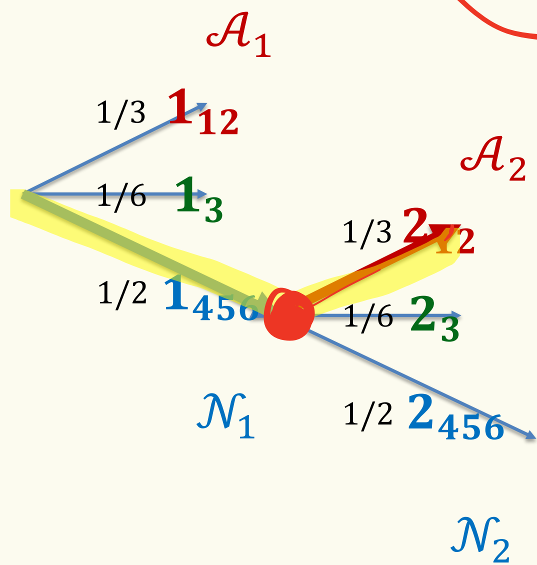
Events:

- $\mathcal{A}_i$ = Alice wins in round i
- $\mathcal{N}_i$ = nobody wins in rounds 1..i

$$\mathbb{P}(\mathcal{A}_2) = \Pr(N_1 \cap A_2)$$

$$= \Pr(N_1)\Pr(A_2 \mid N_1)$$

$$= \frac{1}{2} \cdot \frac{1}{3}$$

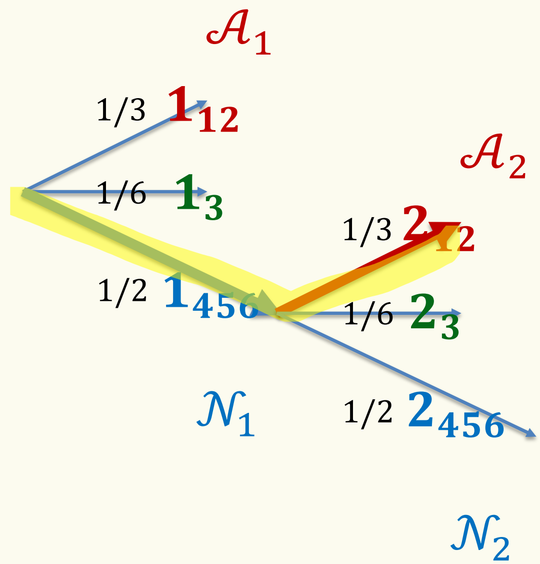


2nd roll indep of 1st roll

Sequential Process – Example

Events:

- $\mathcal{A}_i$ = Alice wins in round i
- $\mathcal{N}_i$ = nobody wins in rounds 1..i



$$\mathbb{P}(\mathcal{A}_2) = \mathcal{P}(\mathcal{N}_1 \cap \mathcal{A}_2)$$
$$= \mathcal{P}(\mathcal{N}_1) \times \mathcal{P}(\mathcal{A}_2 | \mathcal{N}_1)$$
$$= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

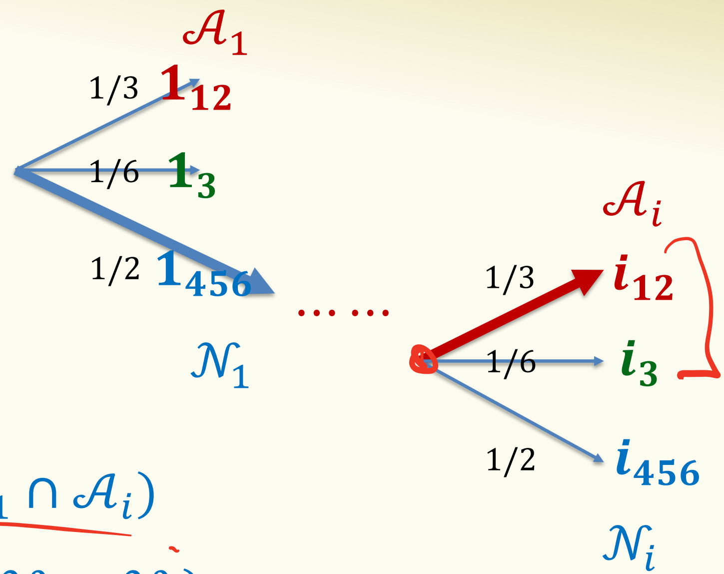
The event $\mathcal{A}_2$ implies $\mathcal{N}_1$, and this means that $\mathcal{A}_2 \cap \mathcal{N}_1 = \mathcal{A}_2$

2^{nd} roll indep of 1^{st} roll

Sequential Process – Example

Events:

- $\mathcal{A}_i$ = Alice wins in round i
- $\mathcal{N}_i$ = nobody wins in round i $1..i$



$$\mathbb{P}(\mathcal{A}_i) = \mathcal{P}(\mathcal{N}_1 \cap \mathcal{N}_2 \cap \cdots \cap \mathcal{N}_{i-1} \cap \mathcal{A}_i)$$

$$= \mathcal{P}(\mathcal{N}_1) \times \mathcal{P}(\mathcal{N}_2|\mathcal{N}_1) \times \mathcal{P}(\mathcal{N}_3|\mathcal{N}_1 \cap \mathcal{N}_2)$$

$$\cdots \times \mathcal{P}(\mathcal{N}_{i-1}|\underbrace{\mathcal{N}_1 \cap \mathcal{N}_2 \cap \cdots \cap \mathcal{N}_{i-1}}_{\mathcal{N}_{i-1}}) \times \mathcal{P}(\mathcal{A}_i|\mathcal{N}_1 \cap \mathcal{N}_2 \cap \cdots \cap \mathcal{N}_{i-1})$$

chain rule

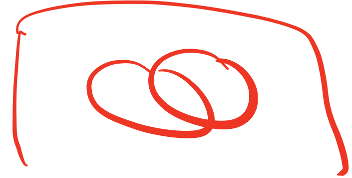
$$= \left(\frac{1}{2}\right)^{i-1} \times \frac{1}{3}$$

Sequential Process -- Example

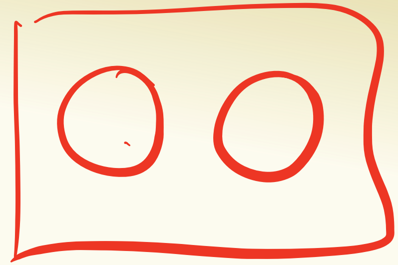
$\mathcal{A}_i$ = Alice wins in round i $\quad \mathbb{P}(\mathcal{A}_i) = \left(\frac{1}{2}\right)^{i-1} \times \frac{1}{3}$

What is the probability that Alice wins?

$$\Pr(\underline{A_1} \cup \underline{A_2} \cup \underline{A_3} \cup \cdots\cdots) = \sum_{i=1}^{\infty} \Pr(A_i) = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^{i-1} \frac{1}{3}$$



Sequential Process -- Example



$\mathcal{A}_i$ = Alice wins in round i $\quad \mathbb{P}(\mathcal{A}_i) = \left(\frac{1}{2}\right)^{i-1} \times \frac{1}{3}$

What is the probability that Alice wins?

$\mathbb{P}(\mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots) = \sum_{i=1}^{\infty} \mathbb{P}(\mathcal{A}_i)$ *All $\mathcal{A}_i$'s are disjoint.*

$$\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^{i-1} \times \frac{1}{3} = \frac{1}{3} \times 2 = \frac{2}{3}$$

Fact. If $|x| < 1$, then $\boxed{\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}}$.

$\frac{1}{3} \quad \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^{i-1}$

$$\boxed{\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{i}} = \frac{1}{1-\frac{1}{2}} = 2$$

Agenda

- A Sequential Process Defined Using Independence
- Independence As An Assumption
- Sometimes Independence Occurs for Nonobvious Reasons
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- Information Cascades

all indep

A, B indep $\Rightarrow$ A, B^c; A^c, B; A^c, B^c

$P(A \cap B) = P(A)P(B)$

Independence as an assumption

- People often assume it **without justification.**
- Example: A sky diver has two chutes

A : event that the main chute doesn't open $\qquad \mathbb{P}(A) = 0.02$

B : event that the backup doesn't open $\qquad \mathbb{P}(B) = 0.1$

- What is the chance that at least one opens assuming independence?

$$\boxed{\begin{aligned} Pr(A \cup B) &= 1 - P(A^c \cap B^c) \\ &= 1 - Pr(A^c)\,P(B^c) \end{aligned}}$$

$$\begin{aligned} Pr(A^c \cap B^c) &= 1 - Pr(A \cup B) \\ &= 1 - (Pr(A) + P(B) - P(A \cap B)) \\ &= 1 - Pr(A) - P(B) + P(A)P(B) \\ &= (1 - Pr(A))(1 - P(B)) \end{aligned}$$

$Pr(A^c \cup B^c) = 1 - P(A \cap B)$
$= 1 - 0.02 \cdot 0.1$

Showed
$\Rightarrow \boxed{Pr(A^c \cap B^c) = P(A^c) P(B^c)}$

Independence as an assumption

- People often assume it **without justification.**
- Example: A sky diver has two chutes

A : event that the main chute doesn't open $\mathbb{P}(A) = 0.02$
B : event that the backup doesn't open $\mathbb{P}(B) = 0.1$

- What is the chance that at least one opens assuming independence?

Assuming independence doesn't justify the assumption! Both chutes could fail because of the same rare event e.g., freezing rain.

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Independence – Another Look

Definition. Two events $\mathcal{A}$ and $\mathcal{B}$ are (statistically) **independent** if

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}).$$

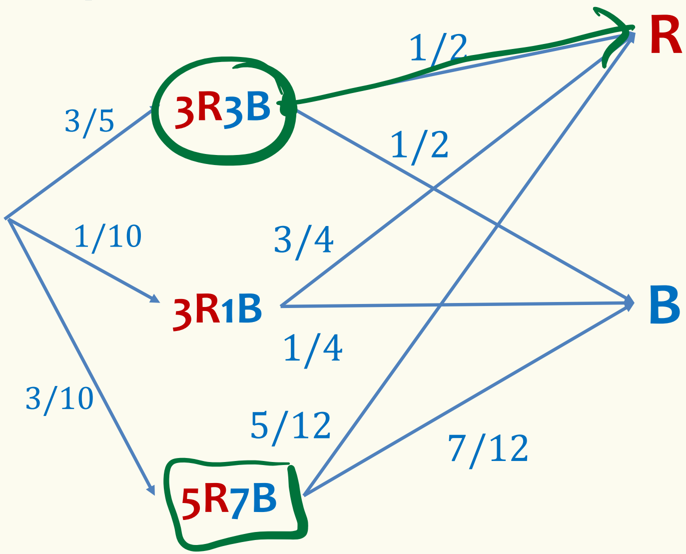
"Equivalently." $\mathbb{P}(\mathcal{A}|\mathcal{B}) = \mathbb{P}(\mathcal{A})$.

Events generated independently ➔ *their probabilities satisfy independence*

~~⬅~~ *Not necessarily*

This can be counterintuitive!

Sequential Process



Setting: An urn contains:

- 3 **red** and 3 **blue** balls w/ probability 3/5
- 3 **red** and 1 **blue** balls w/ probability 1/10
- 5 **red** and 7 **blue** balls w/ probability 3/10

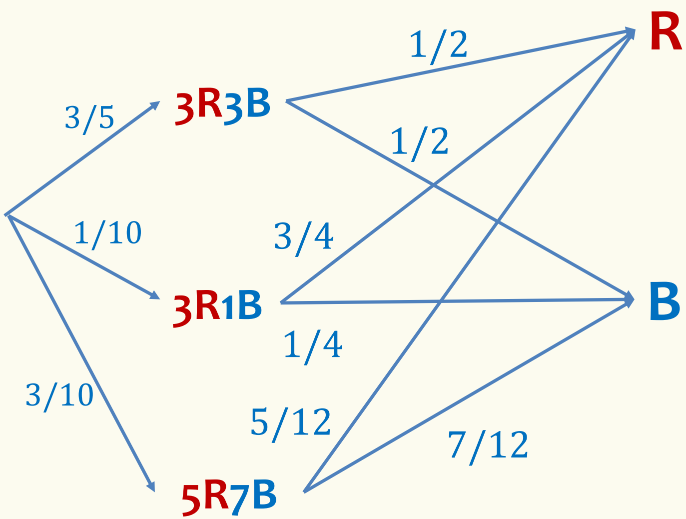
We draw a ball at random from the urn.

Are **R** and **3R3B** independent?

$$P(R) = P(R \mid 3R3B)\,P(3R3B) + P(R \mid 3R1B)\,P(3R1B) + P(R \mid 5R7B)\,P(5R7B)$$

$$= \frac{1}{2} = P(R \mid 3R3B)$$

Sequential Process



Setting: An urn contains:

- 3 **red** and 3 **blue** balls w/ probability 3/5
- 3 **red** and 1 **blue** balls w/ probability 1/10
- 5 **red** and 7 **blue** balls w/ probability 3/10

We draw a ball at random from the urn.

$$\mathbb{P}(\mathbf{R}) = \frac{3}{5}\times\frac{1}{2}+\frac{1}{10}\times\frac{3}{4}+\frac{3}{10}\times\frac{5}{12}=\frac{1}{2}$$

$$\mathbb{P}(\mathbf{3R3B})\times\mathbb{P}(\mathbf{R}\mid\mathbf{3R3B})$$

Are **R** and **3R3B** independent?

Independent! $\mathbb{P}(\mathbf{R}) = \mathbb{P}(\mathbf{R}\mid\mathbf{3R3B})$

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Conditional Independence

Definition. Two events $\mathcal{A}$ and $\mathcal{B}$ are **independent** conditioned on C if $\mathbb{P}(C) \neq 0$ and $\mathbb{P}(\mathcal{A} \cap \mathcal{B} \mid C) = \mathbb{P}(\mathcal{A} \mid C) \cdot \mathbb{P}(\mathcal{B} \mid C)$.

Plain Independence. Two events $\mathcal{A}$ and $\mathcal{B}$ are **independent** if

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}).$$

Equivalence:

- If $\mathbb{P}(\mathcal{A}) \neq 0$, equivalent to $\mathbb{P}(\mathcal{B}|\mathcal{A}) = \mathbb{P}(\mathcal{B})$
- If $\mathbb{P}(\mathcal{B}) \neq 0$, equivalent to $\mathbb{P}(\mathcal{A}|\mathcal{B}) = \mathbb{P}(\mathcal{A})$

Conditional Independence

Definition. Two events $\mathcal{A}$ and $\mathcal{B}$ are **independent** conditioned on C if

$$\mathbb{P}(C) \neq 0 \text{ and } \mathbb{P}(\mathcal{A} \cap \mathcal{B} \mid C) = \mathbb{P}(\mathcal{A} \mid C) \cdot \mathbb{P}(\mathcal{B} \mid C).$$

Equivalence:

- If $\mathbb{P}(\mathcal{A} \cap C) \neq 0$, equivalent to $\boxed{\mathbb{P}(\mathcal{B} | \mathcal{A} \cap C) = \mathbb{P}(B \mid C)}$
- If $\mathbb{P}(\mathcal{B} \cap C) \neq 0$, equivalent to $\mathbb{P}(\mathcal{A} | \mathcal{B} \cap C) = \mathbb{P}(\mathcal{A} \mid C)$

Example – More coin tossing

Suppose there is a coin C1 with Pr(Head) = 0.3 and a coin C2 with Pr(Head) = 0.9. We pick one randomly with equal probability and flip that coin twice independently. What is the probability we get all heads?

$$\Pr(HH) = \Pr(HH \mid C1)\Pr(C1) + \Pr(HH \mid C2)\Pr(C2) \qquad \text{LTP}$$

$$= P(H|C1)\,P(H|C_1)\,P(C1) + \left(P(H|C2)\right)^2 P(C_2)$$

$$= 0.3^2 \tfrac{1}{2} + 0.9^2 \tfrac{1}{2} =$$

Example – More coin tossing

Suppose there is a coin C1 with Pr(Head) = 0.3 and a coin C2 with Pr(Head) = 0.9. We pick one randomly with equal probability and flip that coin 2 times independently. What is the probability we get all heads?

$$\Pr(HH) = \Pr(HH \mid C1)\Pr(C1) + \Pr(HH \mid C2)\Pr(C2)$$ LTP

$$= \Pr(H \mid C2)^2 \Pr(C1) + \Pr(H \mid C2)^2 \Pr(C2)$$ Conditional Independence

$$= 0.3^2 \cdot 0.5 + 0.9^2 \cdot 0.5 = \boxed{0.45}$$

$$\Pr(H) = \Pr(H \mid C1)\Pr(C1) + \Pr(H \mid C2)\Pr(C2) = 0.6$$