CSE 312

Foundations of Computing II

Lecture 8: Random Variables and Expectation

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & Anna ☺

Agenda

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation

Random Variables (Idea)

Often: We want to **capture quantitative properties** of the outcome of a random experiment, e.g.:

- *What is the total of two dice rolls?*
- *What is the number of coin tosses needed to see the first head?*
- *What is the number of heads among 2 coin tosses?*

$\{4, 3\} \to 7$

$\{1, 2\} \to 3$

$HT \to 1$

$\{TH\} \to 2$

$\{TTTTH\} \to 5$

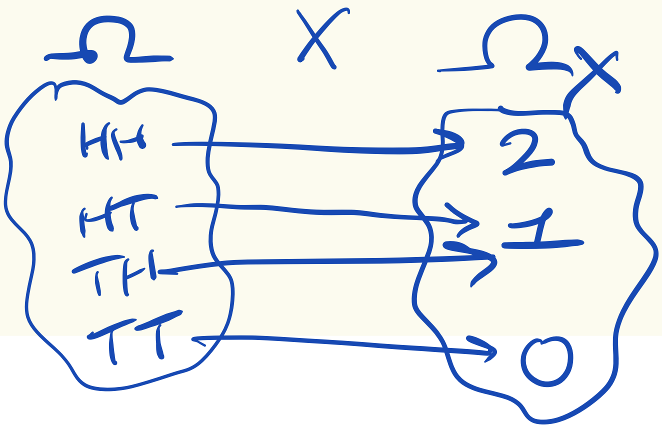
Random Variables

Definition. A **random variable (RV)** for a probability space $(\Omega, \mathbb{P})$ is a function $X : \Omega \to \mathbb{R}$.

Ω_X

The set of values that X can take on is called its range/support Ω_X

Example. Number of heads in 2 independent coin flips $\Omega = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$



RV Example

1 2 ... 20

$X(\{2,7,5\}) \to 7$

$X(\{15,3,8\}) \to 15$

20 balls labeled 1, 2, …, 20 in a ~~bin~~ urn

- Draw a subset of 3 uniformly at random
- Let $X = \text{maximum of the 3 numbers on the balls}$
  - Example: $X(2,7,5) = 7$
  - Example: $X(15,3,8) = 15$
- What is $|\Omega_X|$?

$\{1, 2, 3\}$

Poll: https://pollev.com/~~annakarlin185~~ aleksjovcic835

A. 20^3

B. 20

C. 18

D. $\binom{20}{3}$

E IDK...

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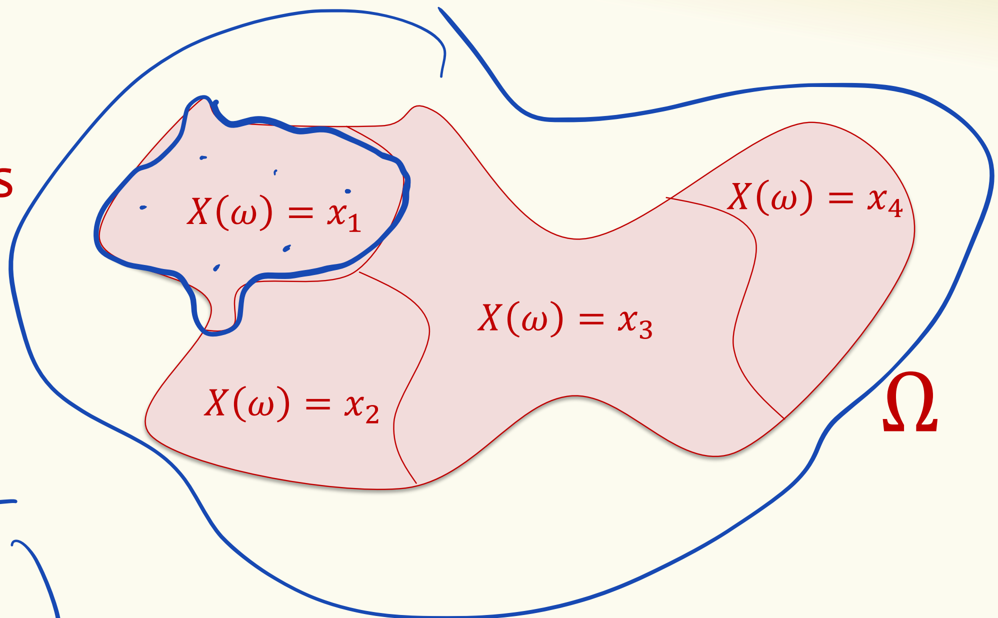
Probability Mass Function (PMF)

Random variables partition the sample space.

$$\sum_{x \in X(\Omega)} \boxed{\mathbb{P}(X = x)} = 1$$

$$\mathbb{P}(X = x)$$



Probability Mass Function (PMF)

ω

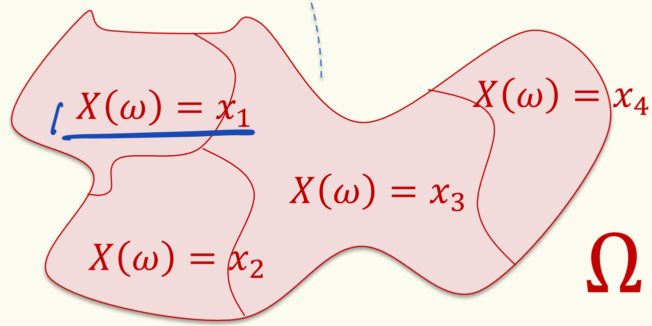
Definition. For a RV $X : \Omega \to \mathbb{R}$, we define the event

$$\{X = x\} \stackrel{\text{def}}{=} \{\omega \in \Omega \mid X(\omega) = x\}$$

We write $\mathbb{P}(X = x) = \mathbb{P}(\{X = x\}) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$ where $\mathbb{P}(X = x)$ is the probability mass function (PMF) of X

Random variables <u>partition</u> the sample space.

$$\sum_{x \in X(\Omega)} \mathbb{P}(X = x) = 1$$



Probability Mass Function (PMF)

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Random variables partition the sample space.

$$\sum_{x \in X(\Omega)} \mathbb{P}(X = x) = 1$$

You also see this notation (e.g. in book):

$$\boxed{\mathbb{P}(X = x)} = \boxed{p_X(x)}$$

$p_X(x)$

Probability Mass Function

Flipping two independent coins

$\Omega = \{HH, \boxed{HT, TH,} TT\}$

X = number of heads in the two flips

$X(HH) = 2$ $\quad X(HT) = 1$ $\quad X(TH) = 1$ $\quad X(TT) = 0$

3

1

$\Omega_X = \{0, 1, 2\}$

$\mathbb{P}(X=2)$

$\hookrightarrow 1/4$

What is $Pr(X = k)$?

$$\mathbb{P}(X=k) = \begin{cases} 1/4 & k=2 \\ 1/2 & k=1 \\ 1/4 & k=0 \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{P}(X=-2)$

$\mathbb{P}(X=0.5)$

RV Example

$\{20, \dots, \dots\}$

20 balls labeled 1, 2, …, 20 in a ~~bin~~ urn

- Draw a subset of 3 uniformly at random
- Let X = maximum of the 3 numbers on the balls

What is $Pr(X = 20)$?

$\mathbb{P}(X=19)$

$$\mathbb{P}(X=20) = \mathbb{P}(\{\dots\}) = \frac{|\{\dots\}|}{|\Omega|} = \frac{\binom{19}{2}}{\binom{20}{3}}$$

Poll:
https://pollev.com/~~annakarlin185~~ aleksjacic835

A. $\binom{20}{2} / \binom{20}{3}$

B. $\binom{19}{2} / \binom{20}{3}$

C. $19^2 / \binom{20}{3}$

D. $19 \cdot 18 / \binom{20}{3}$


gifak-net
h2
h1
b
a
S=a*h1
S=b*h2

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Cumulative Distribution Function (CDF)

Definition. For a RV $X: \Omega \rightarrow \mathbb{R}$, the cumulative distribution function of X specifies for any real number x, the probability that $\boxed{X \leq x}$.

$$\boxed{F_X(x)} = \Pr(X \leq x)$$

CDF $x = -0.5$

Go back to 2 coin clips, where X is the number of heads

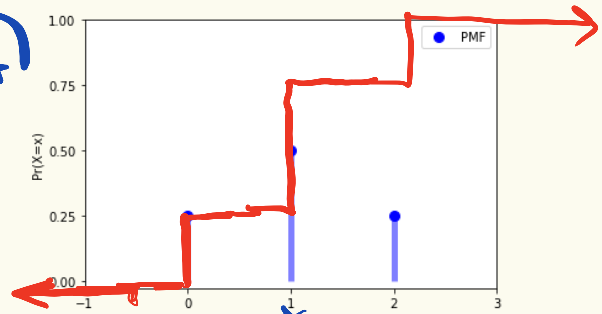
PMF

$\mathbb{P}(X \leq 0)$

$=$

$$\Pr(X = x) = \begin{cases} \frac{1}{4}, & x = 0 \\ \frac{1}{2}, & x = 1 \\ \frac{1}{4}, & x = 2 \end{cases}$$

$\mathbb{P}(X = x)$



X

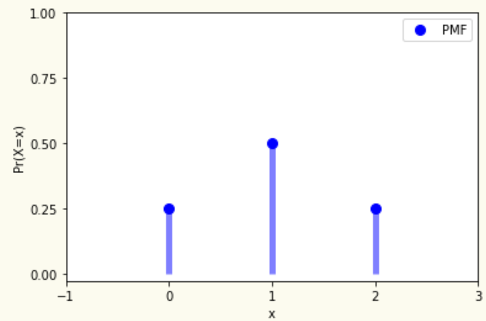
Cumulative Distribution Function (CDF)

Definition. For a RV $X: \Omega \to \mathbb{R}$, the cumulative distribution function of where X specifies for any real number x, the probability that $X \leq x$.

$$\mathrm{F}_X(x) = \Pr(X \leq x)$$

Go back to 2 coin clips, where X is the number of heads

$$\Pr(X = x) = \begin{cases} \frac{1}{4}, & x = 0 \\ \frac{1}{2}, & x = 1 \\ \frac{1}{4}, & x = 2 \end{cases} \qquad F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \leq x < 1 \\ \frac{3}{4}, & 1 \leq x < 2 \\ 1, & 2 \leq x \end{cases}$$



Example: Returning Homeworks

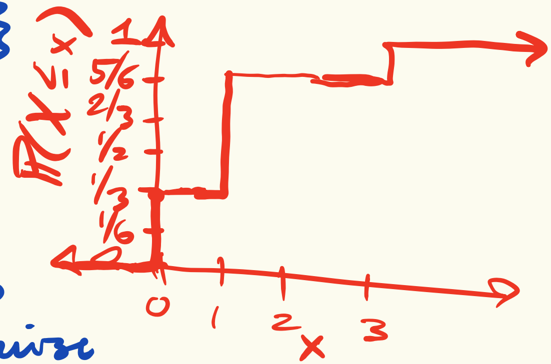
- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$$\Omega_X = \{3, 1, 0\}$$

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

PMF

$$\mathbb{P}(X = x) = \begin{cases} 1/6 & x = 3 \\ 1/2 & x = 1 \\ 1/3 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$



$\mathbb{P}(X \le x)$ $\quad$ $\mathbb{P}(X \le 10)$

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Expectation (Idea)

What is the *expected* number of heads in 2 independent flips of a fair coin?

Expectation

~~Cumulative Disribution Function (CDF)~~

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the expectation or expected value of X is

$$\mathrm{E}[X] = \sum_{\omega \in \Omega} \boxed{X(\omega)} \cdot \Pr(\omega)$$

or equivalently $\mathrm{E}[X]$

$$\mathrm{E}[X] = \sum_{x \in X(\Omega)} x \, \Pr(X = x)$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

Example: Returning Homeworks

$\Omega_X = \{3, 1, 0\}$

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot \mathbb{P}(X = x)$$

$$= 3 \cdot \mathbb{P}(X=3) + 1 \cdot 1/2 + 0 \cdot 1/3 = \boxed{1}$$

$\mathbb{P}(X=3)$: $\cancel{1/2}$ $1/6$

$$= \sum_{\omega \in \Omega} X(\omega)\, \mathbb{P}(\omega)$$

$$= 3 \cdot 1/6 + 1 \cdot 1/6 + 1 \cdot 1/6 + 0 \cdot 1/6 + 0 \cdot 1/6 + 1 \cdot 1/6$$

$$= \boxed{1}$$

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

Flip a Biased Coin Until Heads (Independent Flips)

Suppose a coin has probability p of being heads. Keep flipping independent flips until heads. Let X be the number of flips until heads.

H

TH

What is: $\Pr(X = 1) =$ p

What is: $\Pr(X = 2) =$ $(1-p)p$

$P(X=3) = (1-p)^2 p$

What is: $\Pr(X = k) =$ $(1-p)^{k-1} p$

Flip a Biased Coin Until Heads (Independent Flips)

$$p = \frac{1}{20}$$

Suppose a coin has probability p of being heads. Keep flipping independent flips until heads. Let X be the number of flips until heads. What is $\mathrm{E}[X]$?

$$E[X] = \frac{1}{p}$$

$$E[X] = \sum_{k \in \Omega_X} k \cdot \mathbb{P}(X = k) = \sum_{k=1}^{\infty} k \cdot \mathbb{P}(X = k)$$

$$= \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p = \frac{1}{(1-(1-p))^2} \cdot p = \frac{1}{p^2} \cdot p = \boxed{\frac{1}{p} = E[X]}$$

Geometric Series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad d/dx \text{ both sides}$$

$$\rightarrow \sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$$

sub in w/ $x = (1-p)$

Students on a bus

A group of 120 students are driven on 3 buses to a football game. There are 36 students in the first bus, 40 in the second and 44 in the third. Let Y be the number of students on a uniformly random bus. What is the pmf of Y and E(Y)? When the buses arrive, one of the 120 students is randomly chosen. Let X denote the number of students on the bus of the randomly chosen student. What is the pmf of X and what is E(X)?

Coin flipping again

Suppose we flip a coin with probability p of coming up Heads n times. Let X be the number of Heads in the n coin flips. What is the pmf of X?