CSE 312

Foundations of Computing II

Lecture 9: Linearity of Expectation

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

Last Class:

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Fn (CDF)
- Expectation

Today:

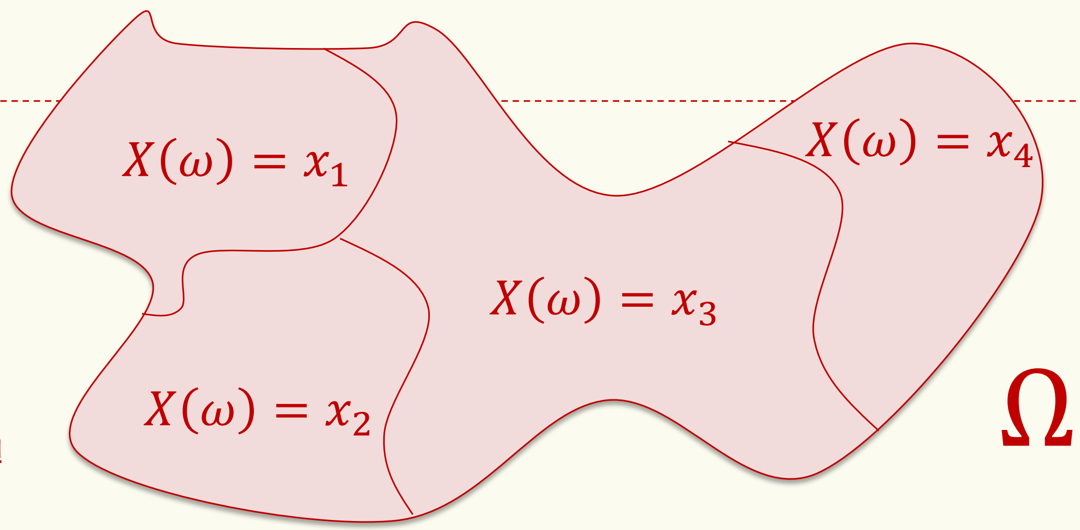
- Recap
- Linearity of Expectation
- Indicator Random Variables

Kandinsky

Reminder: Random Variables

Definition. A **random variable (RV)** defined on a probability space $(\Omega, \mathbb{P})$ is a function $X: \Omega \to \mathbb{R}$.

The set of values that X can take on is called its range/support Ω_X



$\{X = x_i\} \stackrel{\text{def}}{=} \{\omega \in \Omega \mid X(\omega) = x_i\}$

Random variables partition the sample space.

Coin flipping again

Suppose we flip a coin independently n times with probability p of coming up Heads each time. Let the r.v. Z be the number of Heads in the n coin flips.

Probability Mass Function (pmf) and Cumulative Distribution Function (CDF)

Definitions.

For a RV $X: \Omega \to \mathbb{R}$, the probability mass function (pmf) of X specifies for any real number x, the probability that $X = x$.

$$p_X(x) = \Pr(X = x) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$$

$$\sum_{x \in \Omega_X} \mathbb{P}(X = x) = 1$$

For a RV $X: \Omega \to \mathbb{R}$, the cumulative distribution function of X specifies for any real number x, the probability that $X \leq x$.

$$F_X(x) = \Pr(X \leq x)$$

Coin flipping again

Suppose we flip a coin independently n times with probability p of coming up Heads each time. Let the r.v. Z be the number of Heads in the n coin flips. What is the p.m.f. of Z ?

Expectation of Random Variable

Definition. Given a discrete RV $X : \Omega \to \mathbb{R}$, the expectation or expected value of X is

$$\mathrm{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega)$$

or equivalently

$$\mathrm{E}[X] = \sum_{x \in \Omega_X} x \cdot \Pr(X = x)$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

Coin flipping again

Suppose we flip a coin independently n times with probability p of coming up Heads each time. Let the r.v. Z be the number of Heads in the n coin flips. What is the $\mathbb{E}(Z)$?

The brute force method

we flip n coins, each one heads with probability p,

Z is the number of heads, what is $\mathbb{E}(Z)$?

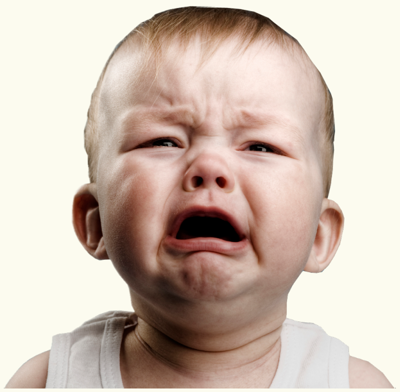
$$\mathbb{E}[Z] = \sum_{k=0}^{n} k \cdot P(Z=k) \quad = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} k \cdot \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} \quad = \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

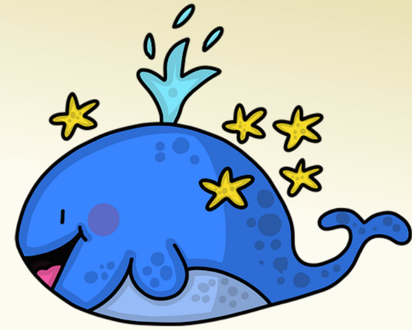
$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!} p^k (1-p)^{(n-1)-k}$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k} = np\big(p + (1-p)\big)^{n-1} = np \cdot 1 = np$$



Linearity of Expectation (Idea)



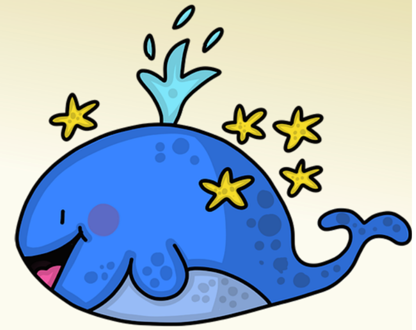
Let's say you and your friend sell fish for a living.

- Every day you catch **X** fish, with **E[X] = 3**.
- Every day your friend catches **Y** fish, with **E[Y] = 7**.

How many fish do the two of you bring in (**Z = X + Y**) on an average day?

E[Z] = E[X + Y] = E[X] + E[Y] = 3 + 7 = 10

Linearity of Expectation (Idea)



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$$E[Z] = E[X + Y] = E[X] + E[Y] = 3 + 7 = 10$$

You can sell each fish for $5 at a store, but you need to pay $20 in rent. How much profit do you expect to make?

$$E[5Z - 20] = 5E[Z] - 20 = 5 \times 10 - 20 = 30$$

Linearity of Expectation – Proof

Theorem. For **any** two random variables X and Y

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

$$\begin{aligned}\mathbb{E}(X+Y) &= \sum_\omega P(\omega)(X(\omega)+Y(\omega)) \\ &= \sum_\omega P(\omega)X(\omega) + \sum_\omega P(\omega)Y(\omega) \\ &= \mathbb{E}(X) + \mathbb{E}(Y)\end{aligned}$$

Linearity of Expectation

Theorem. For **any** two random variables X and Y

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

Or, more generally: For any random variables $X_1, \ldots, X_n$,

$$\mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n).$$

Because: $\mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}((X_1 + \cdots + X_{n-1}) + X_n)$

$$= \mathbb{E}(X_1 + \cdots + X_{n-1}) + \mathbb{E}(X_n) = \cdots$$

Coin flipping again

Suppose we flip a coin independently n times with probability p of coming up Heads each time. Let the r.v. Z be the number of Heads in the n coin flips. What is the $\mathbb{E}(Z)$?

Example – Coin Tosses

we flip n coins, each one heads with probability p

Z is the number of heads, what is $\mathbb{E}(Z)$?

- $X_i = \begin{cases} 1, & i\text{–th coin–flip is heads} \\ 0, & i\text{–th coin–flip is tails.} \end{cases}$

Fact. $Z = X_1 + \cdots + X_n$

Linearity of Expectation:

$$\mathbb{E}(Z) = \mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n) \quad = n \cdot p$$

$\mathbb{P}(X_i = 1) = p$
$\mathbb{P}(X_i = 0) = 1 - p$

$$\mathbb{E}(X_i) = p \cdot 1 + (1 - p) \cdot 0 = p$$

Computing complicated expectations

Often boils down to the following three steps

- Decompose: Finding the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + \cdots + X_n$$

- LOE: Observe linearity of expectation.

$$\mathbb{E}(X) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n).$$

- Conquer: Compute the expectation of each X_i

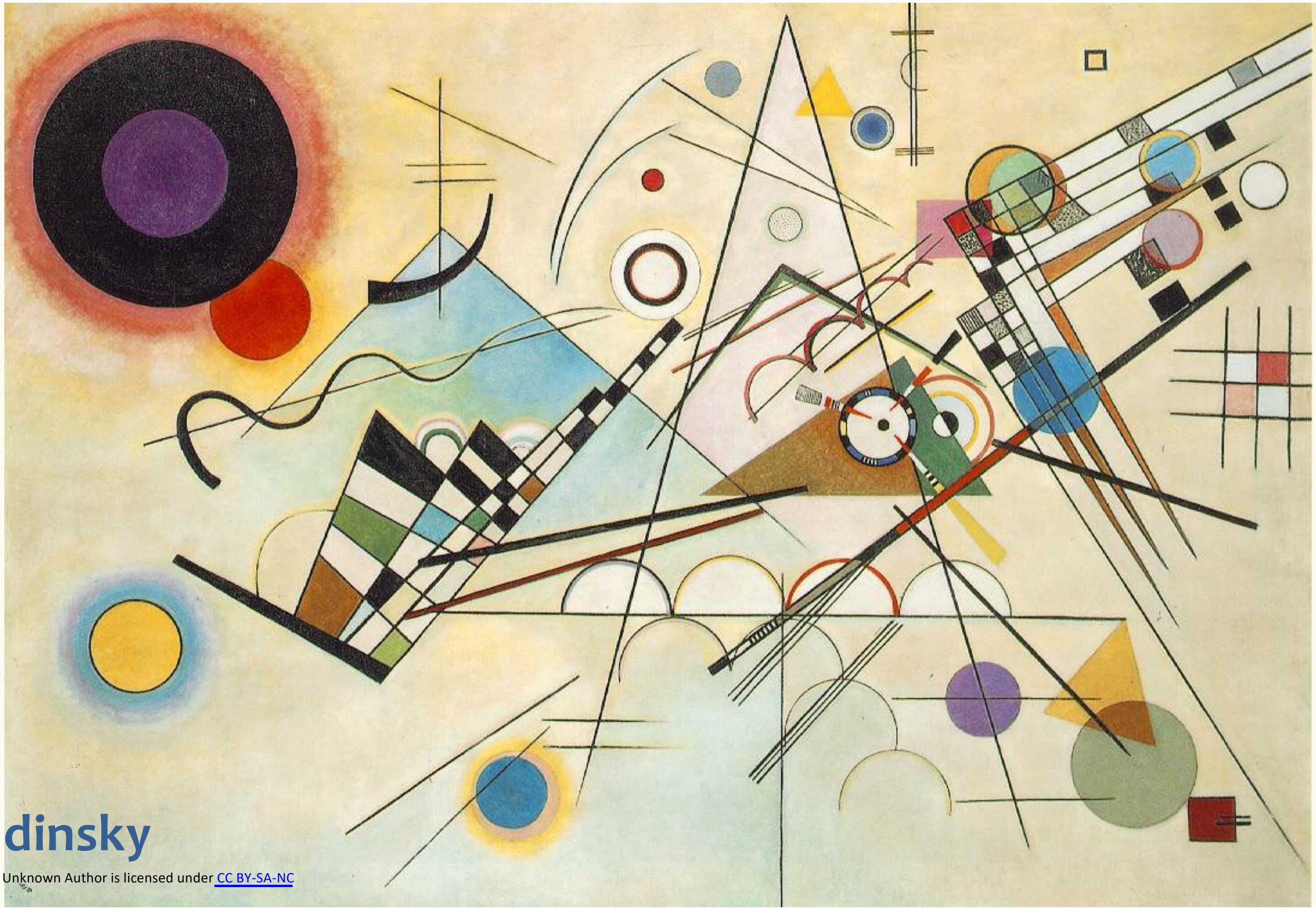
Often, X_i are indicator (0/1) random variables.

Indicator random variable

For any event A, can define the indicator random variable X for A

$$X = \begin{cases} 1 & \text{if event A occurs} \\ 0 & \text{if event A does not occur} \end{cases}$$

$$\mathbb{P}(X = 1) = \mathbb{P}(\mathrm{A})$$
$$\mathbb{P}(X = 0) = 1 - \mathbb{P}(\mathrm{A})$$



Kandinsky

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW
- what is $\mathbb{E}(X)$?

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW
- what is $\mathbb{E}(X)$?
- Use Linearity of Expectation

Decompose: What is X_i?

$\mathbf{Pr}(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

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1/6	3, 1, 2	0
1/6	3, 2, 1	1

Decompose: X_i indicates if student i got their own HW back

LOE:

Conquer: What is $\mathbb{E}(X_i)$? A. $\frac{1}{n}$ B. $\frac{1}{n-1}$ C. ½

https://pollev.com/ annakarlin185

Pairs with same birthday

- In a class of m students, on average how many pairs of people have the same birthday?

Decompose:

LOE:

Conquer:

Linearity of Expectation – Even stronger

Theorem. For any random variables $X_1, \dots, X_n$, and real numbers $a_1, \dots, a_n \in \mathbb{R}$,

$$\mathbb{E}(a_1 X_1 + \cdots + a_n X_n) = a_1 \mathbb{E}(X_1) + \cdots + a_n \mathbb{E}(X_n).$$

Very important: In general, we do not have $\mathbb{E}(X \cdot Y) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$

Linearity is special!

In general $\mathbb{E}(g(X)) \neq g(\mathbb{E}(X))$

E.g., $X = \begin{cases} 1 \ \ with\,prob\ \ 1/2 \\ -1 \ \ with\,prob\ 1/2 \end{cases}$

- $\mathbb{E}(XY) \neq \mathbb{E}(X)\mathbb{E}(Y)$
- $\mathbb{E}(X/Y) \neq \mathbb{E}(X)/\mathbb{E}(Y)$
- $\mathbb{E}(X^2) \neq \mathbb{E}(X)^2$

How DO we compute $\mathbb{E}(g(X))$?

Expectation of $g(X)$

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the expectation or expected value of X is

$$\mathrm{E}[X] = \sum_{\omega \in \Omega} g(X(\omega)) \cdot \Pr(\omega)$$

or equivalently

$$\mathrm{E}[X] = \sum_{x \in X(\Omega)} g(x) \cdot \Pr(X = x)$$

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW
- Let $Y = (X^2+4)\ mod\ 8$.
- what is $\mathbb{E}(Y)$?

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
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Kandinsky