Homework 2: More Counting and Probability

For each problem, remember you must briefly explain/justify how you obtained your answer, as correct answers without an explanation will not receive full credit. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide.

In general, your goal in an explanation is to write enough that a student from class who has attended lecture, but not read the problem yet, could understand your approach, verify your reasoning, and believe your answer is correct. While we do not usually need to see arithmetic, you must include enough work that in principle one could rederive your answer with only a scientific calculator.

Unless a problem states otherwise, you should leave your answer in terms of factorials, combinations, etc., for instance 26^7 or $26!/7!$ or $26 \cdot \binom{26}{7}$ are all good forms for final answers.

Instructions as to how to upload your solutions to gradescope are on the course web page.

Remember that you must tag your written problems on Gradescope.

Submission: You must upload a **pdf** of your written solutions to Gradescope under "HW 2 [Written]". (Instructions as to how to upload your solutions to gradescope are on the course web page.) The use of latex is *highly recommended*. (Note that if you want to hand-write your solutions, you'll need to scan them. We will take off points for hand-written solutions that are difficult to read due to poor handwriting and neatness.)

Due Date: This assignment is due at 11:59 PM Wednesday April 14 (Seattle time, i.e. GMT-7).
You will submit the written problems as a PDF to gradescope. Please put each numbered problem on its own page of the pdf (this will make selecting pages easier when you submit), and ensure that your pdfs are oriented correctly (e.g. not upside-down or sideways).

Collaboration: Please read the full collaboration policy. If you work with others (and you should!), you must still write up your solution independently and name all of your collaborators somewhere on your assignment.

1. Combinatorial Identities [16 points]

Prove each of the following identities using a *combinatorial argument* (i.e., an argument that counts two different ways); an algebraic solution will be marked substantially incorrect.

For the purposes of these problems, using commutativity of multiplication and addition (i.e. $ab = ba$, $a + b = b + a$), and distributivity/factoring ($a(b + c) = ab + ac$) are allowed as part of a combinatorial argument. Any other algebra facts (e.g. Pascal's Rule about combinations, the definition of combinations/permutations in terms of factorials, cancelling numbers that appear in numerators and denominators) would make it an algebraic solution not a combinatorial one.

(a) $\sum_{k=0}^{m} \binom{m}{k}\binom{n}{k} = \binom{m+n}{m}$. You may assume that $n \geq m \geq 0$. Hint: Start with the right hand side and imagine you are choosing a team of m people from a group of people consisting of n Americans and m Canadians.

(b) $\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} = \binom{n}{m}2^{n-m}$. Assume that $n \geq m \geq 0$. Hint: You'll need a different setup for this problem then the last one (i.e., starting with n Americans and m Canadians isn't a good place to start).

2. Working Together [10 points]

Currently, lecture A has 68 students enrolled. Let's say we gave an exam with 7 questions to each of the 68 students, with each student taking the exam individually. It turned out that every question was solved by at least 43 students.

Now, suppose instead we had let you work on the exam in pairs. When you work in pairs if **at least** one of you could get the correct answer, you'd convince your partner and you would both get it right.

Using the pigeonhole principle, show that there exists a pair of students such that, if they work together, they will get a perfect score on the exam.

When using the pigeonhole principle, be sure to mention what the pigeons and pigeonholes are.

3. Stuff into stuff [12 points]

(a) We have 20 (distinguishable) people and 40 (distinguishable) rooms. How many different ways are there to assign the (distinguishable) people to the (distinguishable) rooms? (Any number of people can go into any of the 40 rooms.)

(b) We have 30 identical (indistinguishable) apples. How many different ways are there to place the apples into 20 (distinguishable) boxes? (Any number of apples can go into any of the boxes.)

(c) We have 30 identical (indistinguishable) apples. How many different ways are there to place the apples into 8 (distinguishable) boxes, if each box is required to have at least two apples in it?

4. Sample Spaces and Probabilities [18 points]

For each of the following scenarios first describe the sample space and indicate how big it is (i.e., what its cardinality is) and then answer the question.

(a) You flip a fair coin 50 times. What is the probability of exactly 20 heads?

(b) You roll 2 fair 6-sided dice, one red and one blue. What is the probability that the sum of the two values showing is 4?

(c) You are given a random 5 card poker hand (selected from a single deck). What is the probability you have a full-house (3 cards of one rank and 2 cards of another rank)?

(d) 20 labeled balls are placed into 10 labeled bins (with each placement equally likely). What is the probability that bin 1 contains exactly 3 balls?

(e) There are 30 psychiatrists and 24 psychologists attending a certain conference. Three of these 54 people are randomly chosen to take part in a panel discussion. What is the probability that at least one psychologist is chosen? What is the probability that exactly three psychologists are chosen?

(f) You buy ten cupcakes choosing from 3 different types (chocolate, vanilla and caramel). Cupcakes of the same type are indistinguishable. What is the probability that you have at least one of each type?

5. Miscounting [14 points]

Consider the question: How many **7-card** poker hands (order doesn't matter) are there that contain at least two 3-of-a-kinds (3-of-a-kind means three cards of the same value). For example, this would be a valid hand: ace of hearts, ace of diamonds, ace of spaces, 7 of clubs, 7 of spades, 7 of hearts, and queen of clubs. (Note that a hand consisting of all 4 aces and three of the 7s is also valid.)

Here is how we might compute this:

> To compute the number of hands, apply the product rule. First pick two ranks that have a 3-of-a-kind (e.g. ace and 7 in the example above). For the lower rank of these, pick the suits of the three cards. Then for the higher rank of these, pick the suits of the three cards. Then out of the remaining $52-6=46$ cards, pick one. Therefore there are
>
> $$\binom{13}{2} \cdot \binom{4}{3} \cdot \binom{4}{3} \cdot \binom{46}{1}$$
>
> hands.

In this problem, you will find what is wrong with this solution.

(a) Is there overcounting in the problem? That is a hand which can be produced by multiple outcomes of the sequential process. If there is, give one concrete example of such a hand and two outcomes of the process that produce it. If there is not, briefly (1-2 sentences) explain why there isn't.

(b) Is there undercounting in the problem? That is a hand which cannot be produced by no outcomes of the sequential process. If there is, give one concrete example of such a hand and briefly explain why no outcome produces it. If there is no such hand, briefly explain why all hands are prodcued at least once.

(c) Correct the calculation – in this part you should produce a correct overall formula by subtracting/dividing out any errors that would fit in (a) and adding/multiplying in any errors that would fit in (b).

(d) Find the answer differently – take a different approach to counting this problem (e.g. use a different sequential process). Verify that you get the same number (via a different formula) than the last part.

6. Classic [20 points]

The goals of this problem are to

- Practice induction (remember induction? It's back!) so that you don't totally forget it before you write another inductive proof in 421.
- Realize that while induction works for proving combinatorial identities, it usually leads to longer proofs than other methods.

(a) Use Pascal's Rule to show that for any $n \geq 0$

$$\sum_{k=0}^{n} \binom{n}{k} + \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n+1} \binom{n+1}{k}$$

(you do not need to use induction for this part)

(b) Now, show via induction that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ holds for all natural numbers n. You may use part (a) in your inductive step.

(c) We've seen in lecture that this theorem can be proven very quickly via the binomial theorem. There is also a combinatorial proof (counting subsets of a set of size n, on the left hand side we consider the subsets of size k); of the three versions of this proof (binomial theorem, induction, combinatorial) which do you prefer? Why? (Write 1-3 setences; there are not right or wrong answers).