Section 3: Conditional Probability

Review of Main Concepts

- **Conditional Probability**: $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$
- **Independence**: Events E and F are independent iff $\Pr(E \cap F) = \Pr(E)\Pr(F)$, or equivalently $\Pr(F) = \Pr(F|E)$, or equivalently $\Pr(E) = \Pr(E|F)$
- **Bayes Theorem**: $\Pr(A|B) = \dfrac{\Pr(B|A)\Pr(A)}{\Pr(B)}$
- **Partition:** Nonempty events $E_1, \dots, E_n$ partition the sample space Ω iff
  - $E_1, \dots, E_n$ are exhaustive: $E_1 \cup E_2 \cup \dots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$, and
  - $E_1, \dots, E_n$ are pairwise mutually exclusive: $\forall i \neq j, E_i \cap E_j = \emptyset$
- **Law of Total Probability (LTP)**: Suppose $A_1, \dots, A_n$ partition Ω and let B be any event. Then

  $\Pr(B) = \sum_{i=1}^{n} \Pr(B \cap A_i) = \sum_{i=1}^{n} \Pr(\mathrm{B} \mid A_i)\Pr(A_i)$
- **Bayes Theorem with LTP**: Suppose $A_1, \dots, A_n$ partition Ω and let B be any event. Then $\Pr(A_1|B) = \dfrac{\Pr(B \mid A_1)\Pr(A_1)}{\sum_{i=1}^{n} \Pr(\mathrm{B} \mid A_i)\Pr(A_i)}$. In particular, $\Pr(A|B) = \dfrac{\Pr(B \mid A)\Pr(A)}{\Pr(\mathrm{B} \mid A)\Pr(A) + \Pr(\mathrm{B} \mid A^C)\Pr(A^C)}$
- **Chain Rule**: Suppose $A_1, \dots, A_n$ are events. Then,

$$\Pr(A_1 \cap \dots \cap A_n) = \Pr(A_1)\Pr(A_2|A_1)\Pr(A_3|A_1 \cap A_2)\dots\Pr(A_n|A_1 \cap \dots \cap A_{n-1})$$

1. Random Grades?

Suppose there are three possible teachers for CSE 312: Martin Tompa, Anna Karlin, and Adam Blank. Suppose the probabilities of getting an A in Martin's class is $\frac{5}{15}$, for Anna's class is $\frac{3}{15}$, and for Adam's class is $\frac{1}{15}$. Suppose you are assigned a grade randomly according to the given probabilities when you take a class from one of these professors, irrespective of your performance. Furthermore, suppose Adam teaches your class with probability $\frac{1}{2}$ and Anna and Martin have an equal chance of teaching if Adam isn't. What is the probability you had Adam, given that you received an A? Compare this to the unconditional probability that you had Adam.

2. Marbles in Pockets

Aleks has 5 blue and 3 white marbles in her left pocket, and 4 blue and 4 white marbles in her right pocket. If he transfers a randomly chosen marble from left pocket to right pocket without looking, and then draws a randomly chosen marble from her right pocket, what is the probability that it is blue?

3. Game Show

Corrupted by their power, the judges running the popular game show America's Next Top Mathematician have been taking bribes from many of the contestants. During each of two episodes, a given contestant is either allowed to stay on the show or is kicked off. If the contestant has been bribing the judges, she will be allowed to stay with probability 1. If the contestant has not been bribing the judges, she will be allowed to stay with probability 1/3, independent of what happens in earlier episodes. Suppose that 1/4 of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds.

(a) If you pick a random contestant, what is the probability that she is allowed to stay during the first episode?

(b) If you pick a random contestant, what is the probability that she is allowed to stay during both episodes?

(c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?

(d) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she was bribing the judges?

4. Parallel Systems

A parallel system functions whenever at least one of its components works. Consider a parallel system of n components and suppose that each component works with probability p independently.

(a) What is the probability the system is functioning?

(b) If the system is functioning, what is the probability that component 1 is working?

(c) If the system is functioning and component 2 is working, what is the probability that component 1 is working?

5. Allergy Season

In a certain population, everyone is equally susceptible to colds. The number of colds suffered by each person during each winter season ranges from 0 to 4, with probability 0.2 for each value (see table below). A new cold prevention drug is introduced that, for people for whom the drug is effective, changes the probabilities as shown in the table. Unfortunately, the effects of the drug last only the duration of one winter season, and the drug is only effective in 20% of people, independently.

number of colds	no drug or ineffective	drug effective
0	0.2	0.4
1	0.2	0.3
2	0.2	0.2
3	0.2	0.1
4	0.2	0.0

(a) Sneezy decides to take the drug. Given that he gets 1 cold that winter, what is the probability that the drug is effective for Sneezy?

(b) The next year he takes the drug again. Given that he gets 2 colds in this winter, what is the updated probability that the drug is effective for Sneezy?

(c) Why is the answer to (b) the same as the answer to (a)?

6. A game

Howard and Jerome are playing the following game: A 6-sided die is thrown and each time it's thrown, regardless of the history, it is equally likely to show any of the six numbers.

- If it shows 5, Howard wins.
- If it shows 1, 2, or 6, Jerome wins.
- Otherwise, they play a second round and so on.

What is the probability that Jerome wins on the 4th round?

7. Another game

Alice and Alicia are playing a tournament in which they stop as soon as one of them wins n games. Alicia wins each game with probability p and Alice wins with probability $1-p$, independently of other games. What is the probability that Alicia wins and that when the match is over, Alice has won k games?

8. Dependent Dice Duo

This problem demonstrates that independence can be "broken" by conditioning. Let D_1 and D_2 be the outcomes of two independent rolls of a fair die. Let E be the event "$D_1 = 1$", F be the event "$D_2 = 6$", and G be the event "$D_1 + D_2 = 7$". Even though E and F are independent, show that

$$\mathbb{P}(E \cap F \mid G) \neq \mathbb{P}(E \mid G)\, \mathbb{P}(F \mid G).$$

9. Infinite Lottery

Suppose we randomly generate a number from the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$. Let A_k be the event we generate the number k, and suppose $\Pr(A_k) = (\frac{1}{2})^k$. Once we generate a number k, that is the maximum we can win. That is, after generating a value k, we can win any number in $[k] = \{1, ..., k\}$ dollars. Suppose the probability that we win $\$j$ for $j \in [k]$ is "uniform", that is, each has probability $\frac{1}{k}$. Let B be the event we win exactly $\$1$. Given that we win exactly one dollar, what is the probability that the number generated was also 1? You may use the fact that $\sum_{j=1}^{\infty} \frac{1}{j \cdot a^j} = \ln(\frac{a}{a-1})$for $a > 1$.

10. The Monty Hall Problem

The Monty Hall problem is a famous, seemingly counter-intuitive probability puzzle named after Monty Hall, the host of the show "Let's Make a Deal". This problem emphasizes the importance of using given information to make decisions.

Assume you are a contestant on this game show. In the original problem, there are 3 doors, one hiding a car and the other two hiding goats. At first, you randomly pick a door, hoping you can win the car. As Monty knows exactly what door hides the location of the car, he purposefully decides to reveal a door different from your pick which is guaranteed to reveal a goat. As there are 2 doors left, Monty later asks if you want to stick to your current door or to switch to the other door.

In the beginning, when there is no information about these 3 doors, every door has equal probability of revealing a car. However, after knowing that Monty will only open a door which definitely reveals a goat, it turns out that switching to the other door yields a higher probability of winning than sticking to your current door. Thus, the best strategy is to switch to the other door. Feel free to do any calculations on your own to find out why.

For this problem, you have to determine the best strategy when there are 4 doors. As a contestant, you first randomly choose a door. Monty opens one of the 3 other doors, which reveals a goat, and asks if you want to stick to your current choice or switch to a different door. After you make your pick, Monty opens another door (other than your current pick) which also reveals a goat. This time, you have to make the final pick: sticking to the current door in the previous pick or switching to the other door. Make a thorough analysis of all possible strategies and explain which one is the best.