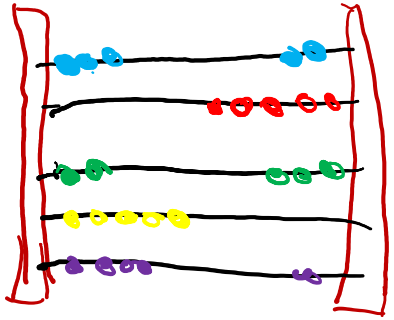
Download the activity pdf while you're waiting



More Counting

CSE 312 Spring 21
Lecture 2

Announcements

Syllabus is up! Thank you for your patience.

Please read the collaboration policy, in particular:
on homeworks you are **encouraged** to collaborate, but the writeup must be your own.

If you're working with others, don't take notes from your discussions

And take a 30 minute break between discussions and your writeup

The goal of these rules is to make sure you've learned how to do the problems.

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Announcements

Homework 1 will come out tonight.

Mostly written problems.

Every written problem requires a **justification.**

Not a proof (unless we say to prove something). But enough explanation that someone who followed lecture but hasn't seen the problem would fully understand where your answer came from (and believe it's correct)

There's also one programming question – submission on gradescope, but it will be easiest to do the questions on Ed (Ed has a python interpreter built in, you could set up your own python environment, but Ed will be sufficient for this quarter).

Announcements

You’ll have 6 late days to use for the quarter.

At most 3 late days per assignment.

Late days are for “normal” things during the quarter

If you have an unusual or extended or extreme issue, please let us know.

The sooner you let us know, the more options we have for accommodations.

Where Are We?

Last time:

Sum and Product Rules

Sequential Processes

Representation is important!

Today:

Combinations and Permutations.

More sequence practice

How many length 3 sequences are there consisting of distinct elements of $\{1,2,3\}$.

Pause

Questions in combinatorics and probability are often dense. A single word can totally change the answer. Does order matter or not? Are repeats allowed or not? What makes two things "count the same" or "count as different"?

Let's look for some keywords

How many length 3 sequences are there consisting of distinct elements of $\{1,2,3\}$.

Sequences implies that order matters – (1,2,3) and (2,1,3) are different.
Distinct implies that you can't repeat elements (1,2,1) doesn't count.

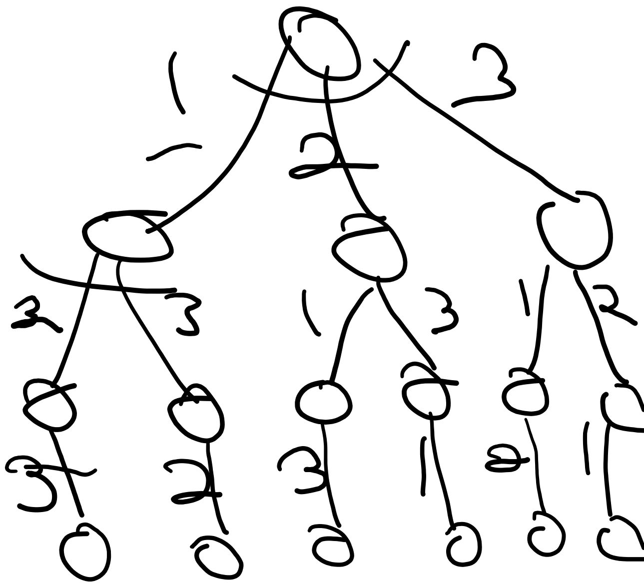
$\{1,2,3\}$ is our "universe" – our set of allowed elements.

More sequence practice

How many length 3 sequences are there consisting of distinct elements of $\{1,2,3\}$.

Step 1:

Step 2:



Step 1: 3 options for the first element.

Step 2: 2 (remaining) options for the second element.

Step 3: 1 (remaining) option for the third element.

$3 \cdot 2 \cdot 1. = 6.$

Factorial

That formula shows up a lot.

The number of ways to "permute" (i.e. "reorder", i.e. "list without repeats") n elements is "n factorial"

n factorial

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 1$$

We only define $n!$ for natural numbers n.

As a convention, we define: $0! = 1$.

Distinct Letters

26

How many length 5 strings are there over the alphabet $\{a, b, \ldots, z\}$ where each string does not repeat a letter.

E.g. "azure" is an allowed string, but "steve" is not, nor is "abcdef"

zarue

$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22$

$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22$

In General

$n = 26$, $k = 5$ $P(26, 5)$

k-permutation

The number of k-element sequences of distinct symbols from a universe of n symbols is:

$$P(n,k) = n \cdot (n-1) \cdots (n-k+1) \cdot (n-k)! = \boxed{\frac{n!}{(n-k)!}}$$

$(n-k)!$

Said out loud as "P n k" or "n permute k"

Alternative notation: ${}_nP_k$

Edge cases: $\cancel{P(n,n) = 1}$, $P(n,0) = 1$, $P(n,k)$ for $k < 0$ or $k > n$ is undefined.

$P(n,n) = n!$, $P(n,1) = n$

Change it slightly

How many **subsets** of size 5 are there of $\{a, b, \dots, z\}$

Remember subsets we don't count repeats – so we still have that rule.

But for subsets order doesn't matter.

$\{a, z, u, r, e\}$ is the same set as $\{a, u, r, e, z\}$ (even though "azure" and "aurez" are different strings).

Clever approach – count two ways

Let's artificially introduce a requirement that we are supposed to have an ordered list.

Then the total is going to be $P(26,5)$.

How else could we get an ordered list? With this sequential process:

Step 1: Choose a subset. → ???

Step 2: Put the subset in order. → 5!

These better give us the same number, so:

$$\frac{26!}{(26-5)!} = ? \cdot 5!$$

Clever approach – count two ways

Let's artificially introduce a requirement that we are supposed to have an ordered list.

Then the total is going to be $P(26,5)$.

How else could we get an ordered list? With this sequential process:

Step 1: Choose a subset.

Step 2: Put the subset in order.

These better give us the same number, so:

$$\frac{26!}{(26-5)!} =?\cdot 5!$$

So the number of size-5 subsets of a size-26 set is:

$$\left(\frac{26!}{(26-5)!\,5!} \right)$$

Number of Subsets

k-combination

The number of k-element subsets from a set of n symbols is:

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!\,(n-k)!}$$

Said out loud "n choose k" (or sometimes: "n combination k")

Lots of notation:

$_nC_k$ or $\binom{n}{k}$ or $C(n,k)$ all mean "number of size-k subsets of a size-n set."

Edge cases: $\binom{n}{0} = 1, \binom{n}{n} = 1$; $\binom{n}{k}$ for $k < 0$ or $k > n$ is undefined.

Second Takeaway

a z x y w | v . . . . b

The second way of counting hints at a generally useful trick:

Pretend that order does matter, then divide by the number of orderings of the parts where order doesn't matter.

For example, here's another way to get the formula for combinations:

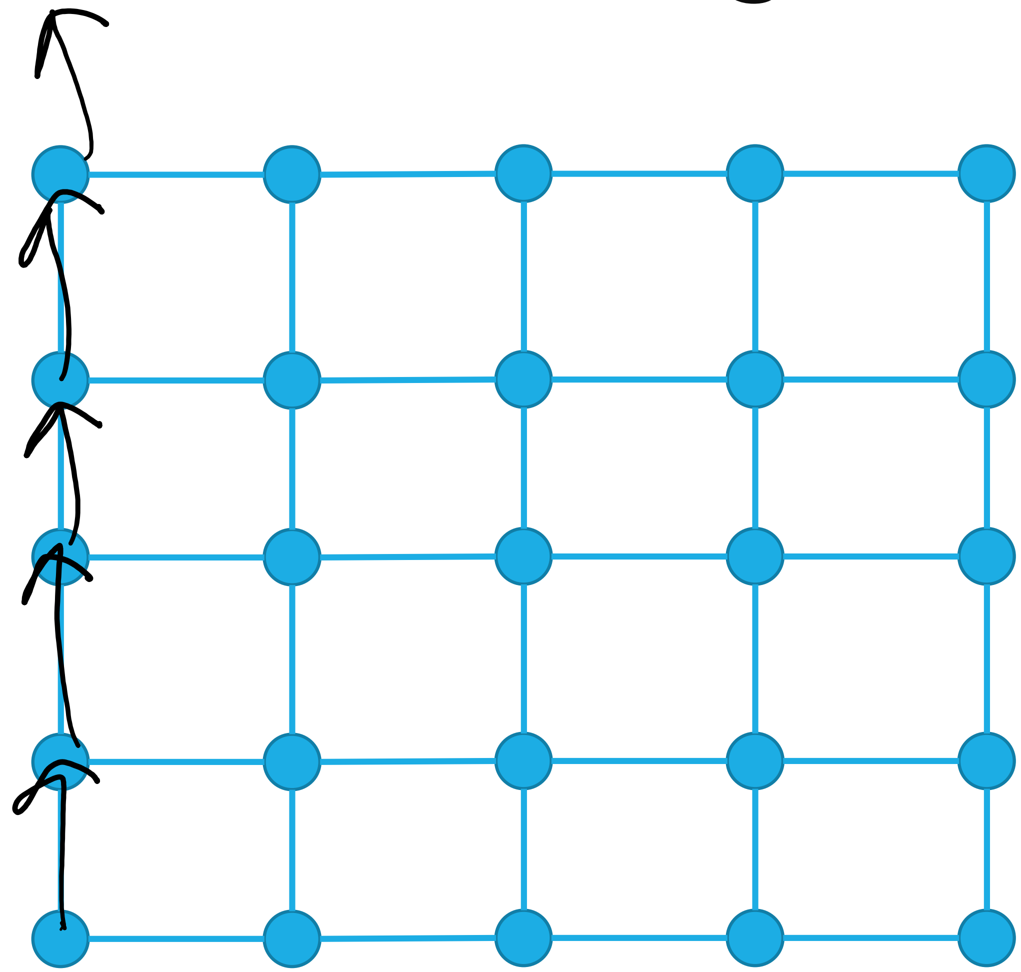
You have n elements. Put them in order, take the first k as your set.

$n!$ Orderings overall. We've overcounted because:

- Among the first k, order doesn't matter between them. Divide by $k!$.
- Among the last $n-k$, order doesn't matter between them. Divide by $(n-k)!$.

$$\frac{n!}{k!\,(n-k)!}$$

Path Counting



We're in the lower-left corner, and want to get to the upper-right corner.

We're only going to go right and up.

How many different paths are there?

$A.\ 2^8$

$B.\ P(8,4)$

$C.\ \binom{8}{4}$

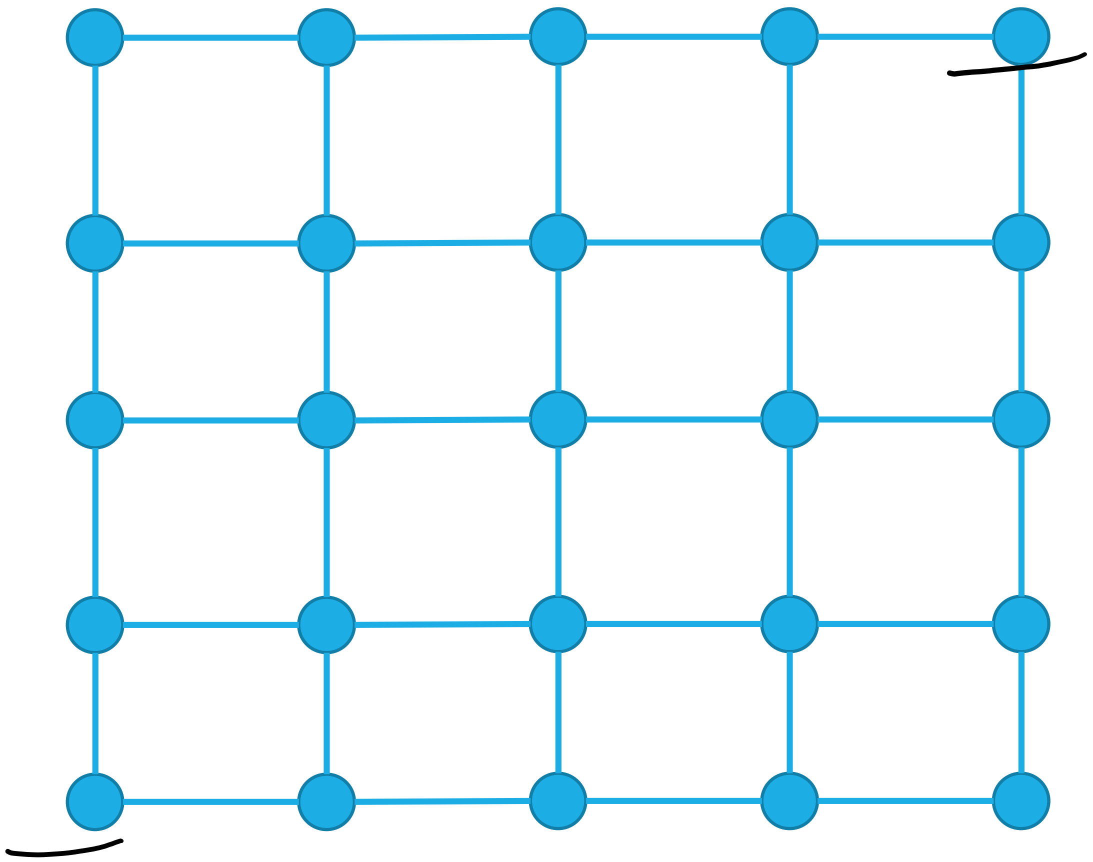
D. Something else

Fill out the poll everywhere so Robbie knows how much to explain

cse312

Go to pollev.com/[redacted] and login with your UW identity

Path Counting



We're in the lower-left corner, and want to get to the upper-right corner.

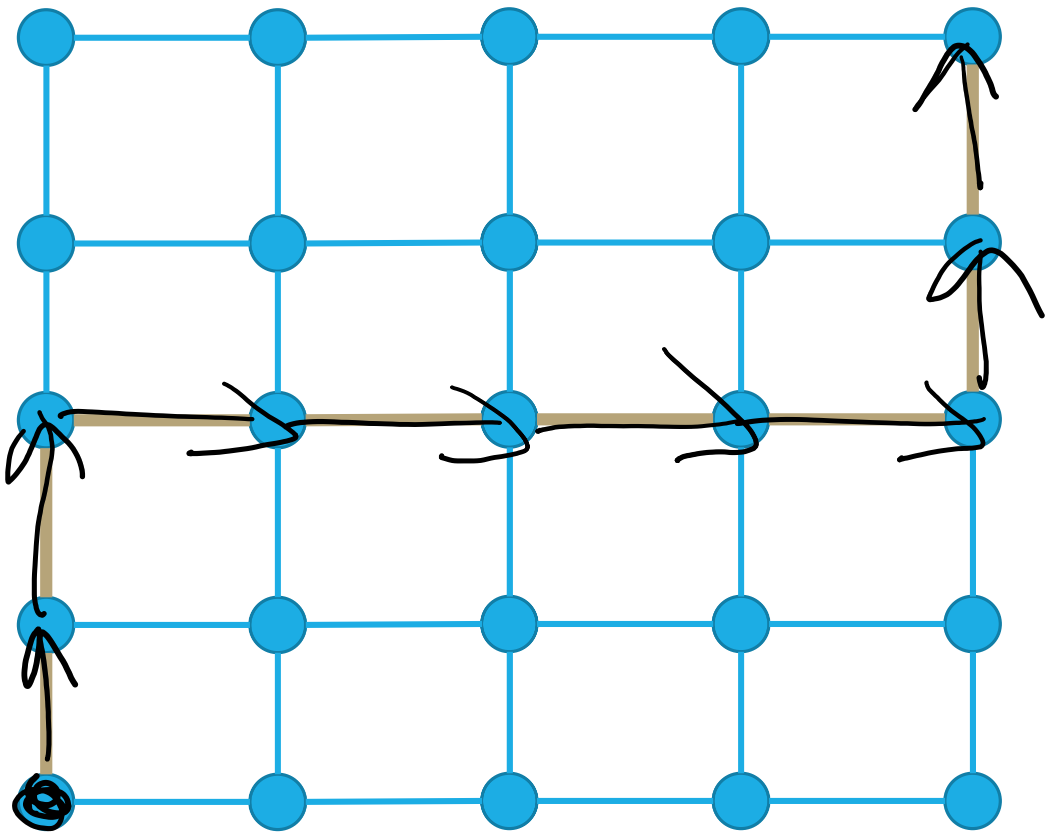
We're only going to go right and up.

How many different paths are there?

Idea 1: $\{1, 2, 3, 4, 5, 6, 7, 8\}$

We're going to take 8 steps.
Choose which SET of 4 of the steps will be up (the others will be ~~down~~ right).

Path Counting



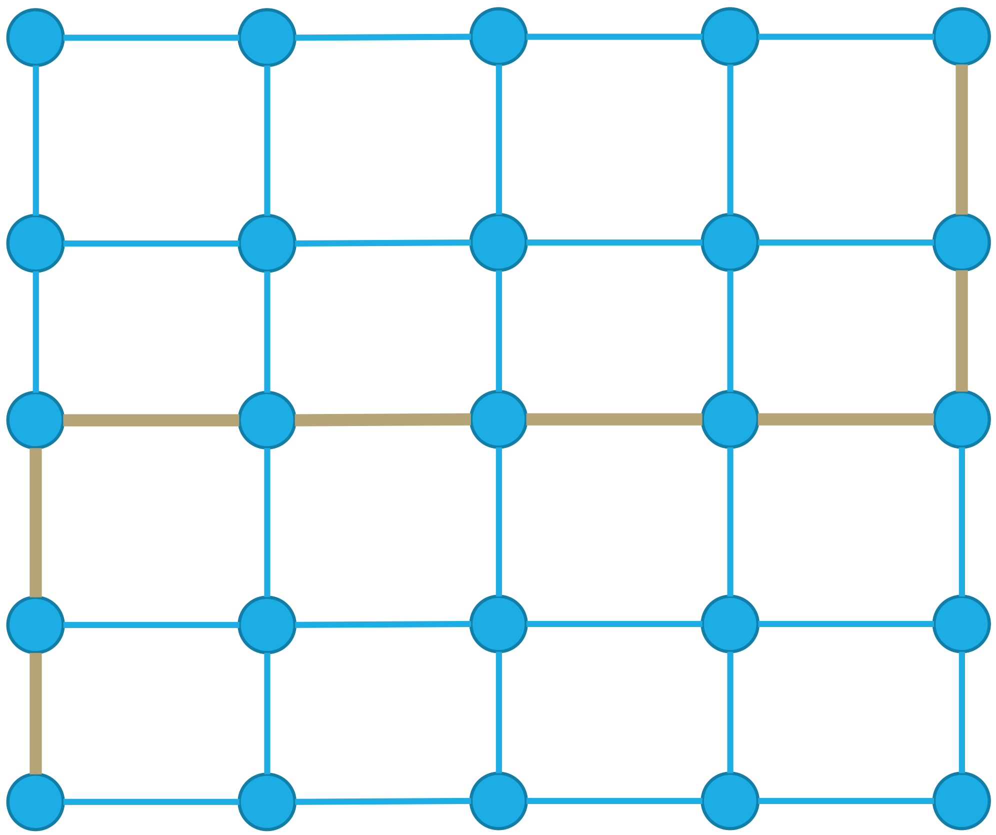
We're in the lower-left corner, and want to get to the upper-right corner.

We're only going to go right and up.

How many different paths are there?

Idea 1:
We're going to take 8 steps.
Choose which SET of 4 of the steps will be up (the others will be down).
E.g. {1,2,7,8} is 8, 1, 2, 7
How many size-4 subsets of {1,2,3,4,5,6,7,8} are there? $\binom{8}{4}$

Path Counting



We’re in the lower-left corner, and want to get to the upper-right corner.

We’re only going to go right and up.

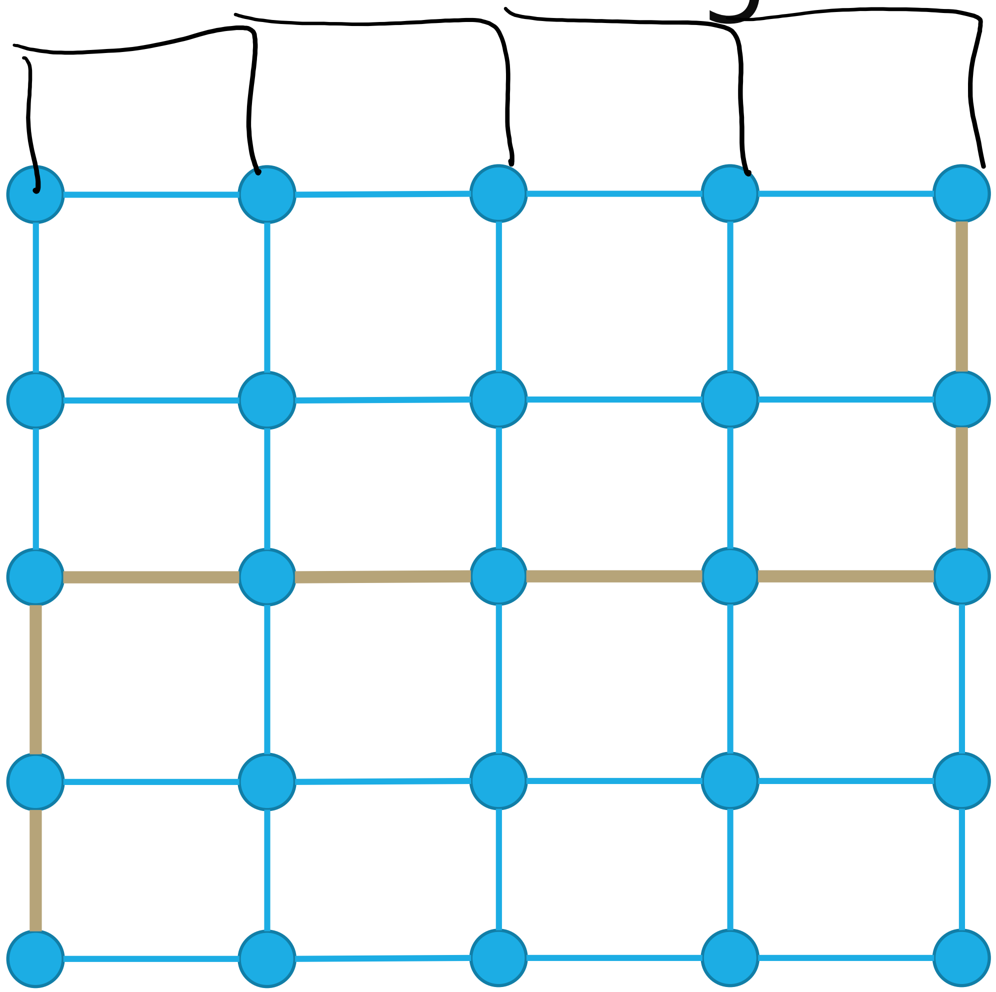
How many different paths are there?

Idea 1:
We’re going to take ~~8~~ 9 steps.
Choose which SET of 4 of the steps will be up (the others will be down).
E.g. $\{1,2,7,8\}$ is
How many size-4 subsets of $\{1,2,3,4,5,6,7,8\}$ are there?

$\binom{9}{4}$

$\binom{8}{4}$ is the answer.

Path Counting



We're in the lower-left corner, and want to get to the upper-right corner.

We're only going to go right and up.

How many different paths are there?

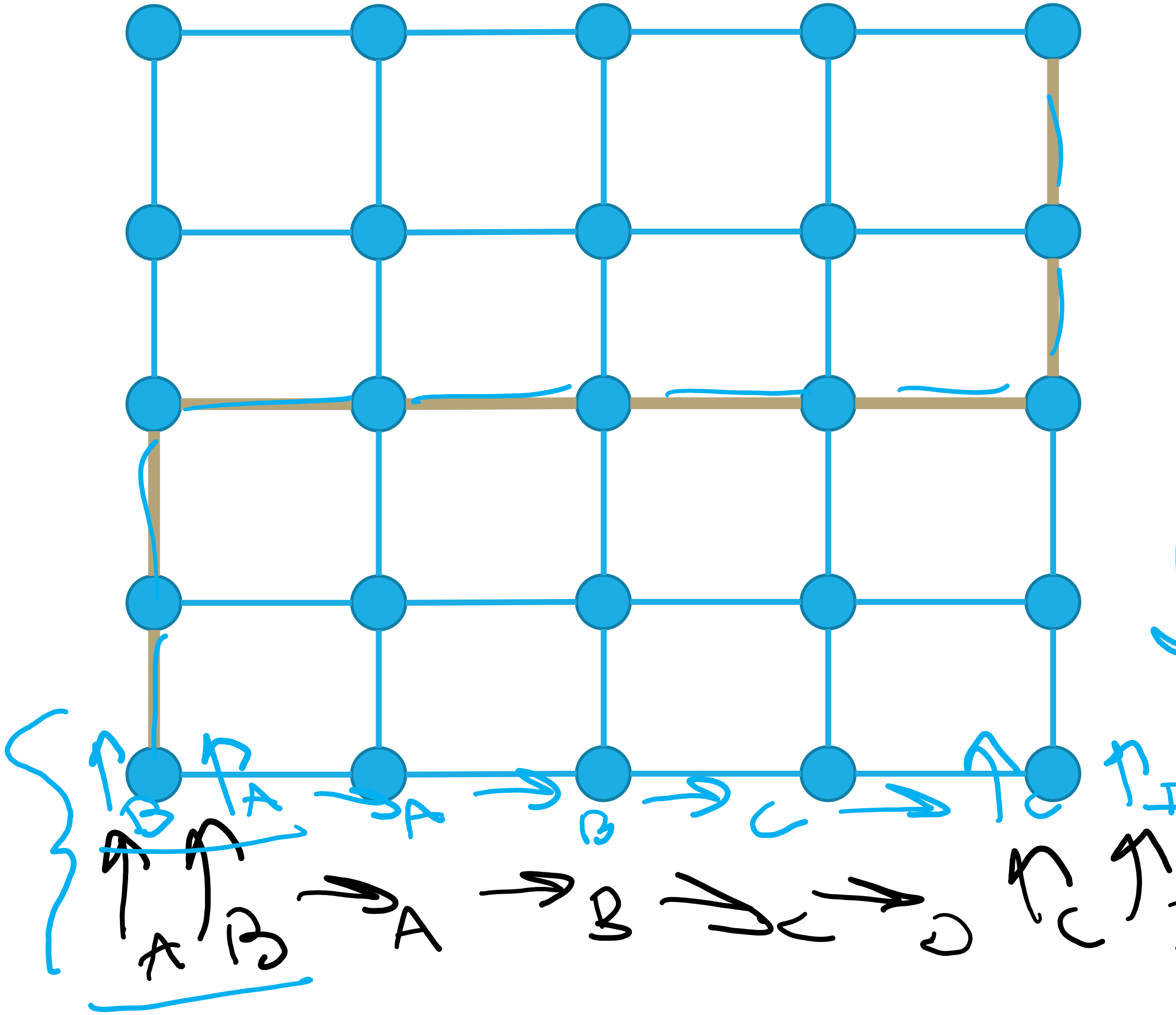
Idea 2: Introduce artificial ordering

Order $\underbrace{\uparrow_A \uparrow_B \uparrow_C \uparrow_D \uparrow_E}_{5!} \underbrace{\rightarrow_A \rightarrow_B \rightarrow_C \rightarrow_D}_{4!}$ $9!$,

$$\binom{9}{5}$$

Path Counting

$\binom{9}{4} = \binom{9}{5}$



We're in the lower-left corner, and want to get to the upper-right corner.

We're only going to go right and up.

How many different paths are there?

Idea 2: Introduce artificial ordering

Order $\uparrow_A\uparrow_B\uparrow_C\uparrow_D\rightarrow_A\rightarrow_B\rightarrow_C\rightarrow_D$ $\quad 8!$

Remove the overcounting

Those 4 $\uparrow$ are really the same, divide by $4!$

The 4 $\rightarrow$ are really the same, divide by $4!$

Total: $\frac{8!}{4!\cdot 4!} \rightarrow \frac{8!}{(8-4)!\,4!} = \binom{8}{4}$.

$\binom{8}{4}$ is the answer.

Overcounting

How many anagrams are there of SEATTLE

(an anagram is a rearrangement of letters).

It's not 7! That counts SEATTLE and SEATTLE as different things!

I swapped the Es (or maybe the Ts)

Overcounting

How many anagrams are there of SEATTLE

Pretend the order of the Es (and Ts) relative to each other matter (that SEATTLE and SEATTLE are different)

How many arrangements of SEATTLE? $7!$

How have we overcounted? Es relative to each other and Ts relative to each other $2! \cdot 2!$

Final answer $\frac{7!}{2! \cdot 2!}$

One More Counting Technique

Complementary Counting

Count the complement of the set you're interested in.

How many length 5 strings over $\{a, b, c, \ldots, z\}$ are there with **at least** 1 'a'

Let A be the set of strings we're interested in, $\mathcal{U}$ be all length 5 strings

$|A| = |\mathcal{U} \setminus \bar{A}| = |\mathcal{U}| - |\bar{A}| = 26^5 - 25^5$

Combination Facts

Some Facts about combinations

Symmetry of combinations: $\binom{n}{k} = \binom{n}{n-k}$

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Two Proofs of Symmetry

Proof 1: By algebra

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ Definition of Combination

$= \frac{n!}{(n-k)!k!}$ Algebra (commutativity of multiplication)

$= \binom{n}{n-k}$ Definition of Combination

Two Proofs of Symmetry

Wasn’t that a great proof.

Airtight. No disputing it.

Got to say “commutativity of multiplication.”

But…do you know *why*? Can you *feel* why it’s true?

Two Proofs of Symmetry

Suppose you have n people, and need to choose k people to be on your team. We will count the number of possible teams two different ways.

Way 1: We choose the k people to be on the team. Since order doesn't matter (you're on the team or not), there are $\binom{n}{k}$ possible teams.

Way 2: We choose the $n-k$ people to NOT be on the team. Everyone else is on it. Since order again doesn't matter, there are $\binom{n}{n-k}$ possible ways to choose the team.

Since we're counting the same thing, the numbers must be equal.
So $\binom{n}{k} = \binom{n}{n-k}$.

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$
\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-1-[k-1])!} + \frac{(n-1)!}{k!(n-1-k)!} && \text{definition of combination} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} && \text{subtraction} \\
&= \frac{[(n-1)!k!(n-k-1)!] + [(n-1)!(k-1)!(n-k)!]}{k!(k-1)!(n-k)!(n-k-1)!} && \text{Find a common denominator} \\
&= \frac{(n-1)!(k-1)!(n-k-1)!\,[k+(n-k)]}{k!(k-1)!(n-k)!(n-k-1)!} && \text{factor out common terms} \\
&= \frac{(n-1)!\,[k+(n-k)]}{k!(n-k)!} && \text{Cancel } (k-1)!(n-k-1)! \\
&= \frac{(n-1)! \cdot n}{k!(n-k)!} = \frac{n!}{k!(n-k)!} && \text{Algebra} \\
&= \binom{n}{k} && \text{Definition of combination}
\end{aligned}
$$

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

You and $n-1$ other people are trying out for a k person team. How many possible teams are there?

Way 1: There are n people total, of which we're choosing k (and since it's a team order doesn't matter) $\binom{n}{k}$.

Way 2: There are two types of teams. Those for which you make the team, and those for which you don't.

If you do make the team, then $k-1$ of the other $n-1$ also make it.

If you don't make the team, k of the other $n-1$ also make it.

Overall, by sum rule, $\binom{n-1}{k-1} + \binom{n-1}{k}$.

Since we're computing the same number two different ways, they must be equal. So: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Takeaways

Formulas for factorial, permutations, combinations.

A useful trick for counting is to pretend order matters, then account for the overcounting at the end (by dividing out repetitions)

When trying to prove facts about counting, try to have each side of the equation count the same thing.

Much more fun and much more informative than just churning through algebra.

Extra Practice

Books, revisited

Remember the books problem from lecture 1? Books 1,2,3,4,5 need to be assigned to Alice, Bob, and Charlie (each book to exactly one person).

Now that we know combinations, try a sequential process approach. It won't be as nice as the change of perspective, but we can make it work.

Break into cases based on how many books Alice gets, use the sum rule to combine.

Books, revisited

Step 1: give Alice gets 0 books (1 way to do this)

Step 2: give Bob a subset of the remaining books 2^5 ways.

Step 3: give Charlie the remaining books (no choice – 1 way)

$+$

Step 1: give Alice 1 book ($\binom{5}{1}$ ways to do this)

Step 2: give Bob a subset of the 4 remaining books 2^4 ways.

Step 3: give Charlie the remaining books (no choice – 1 way)

$+ \ \dots$

Books, revisited

Add all the options together

$$1 \cdot 2^5 \cdot 1 + \binom{5}{1} \cdot 2^4 \cdot 1 + \binom{5}{2} \cdot 2^3 \cdot 1 + \binom{5}{3} \cdot 2^2 \cdot 1 + \binom{5}{4} \cdot 2^1 \cdot 1 + \binom{5}{5} \cdot 2^0 \cdot 1$$

If you plug and chug, you'll get the number we got last time. It took quite a bit of work, but we got there!