No activity slide today

Even More Counting

CSE 312 Spring 21
Lecture 3

Announcements

HW 1 is out! Due Wednesday.

Office hours start today! Visit now before others start on the homework.

Also, please start early.

One thing we haven’t covered yet – a “hand” of cards is a set of cards.

I.e. it’s not ordered.

Announcements

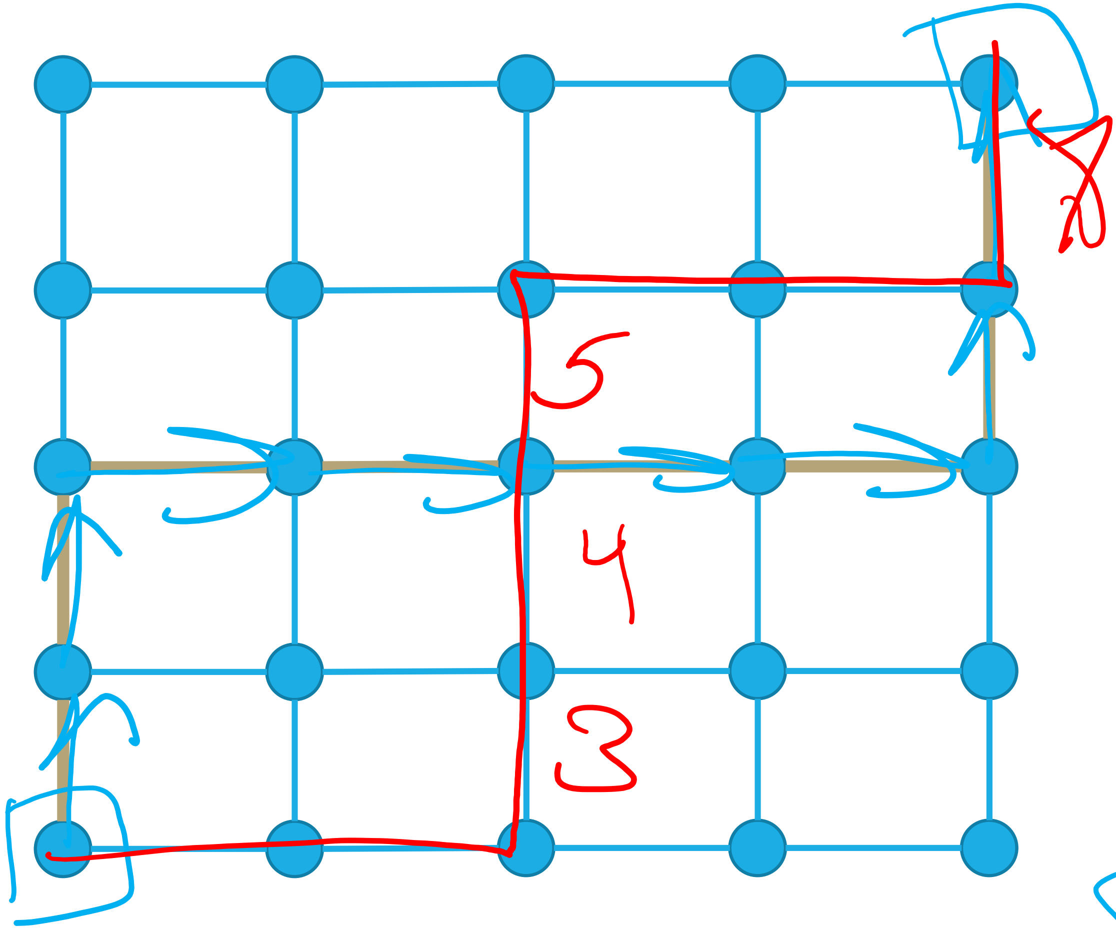
There’s an optional (online) textbook.

It’s linked on the webpage, with the relevant sections on the calendar.

Useful if you want a different perspective.

Occasional differences in notation/vocabulary, but still useful!

Path Counting



We're in the lower-left corner, and want to get to the upper-right corner.

We're only going to go right and up.

How many different paths are there?

Idea 1:
We're going to take 8 steps.
Choose which SET of 4 of the steps will be up (the others will be down).
E.g. $\{1,2,7,8\}$ is
How many size-4 subsets of $\{1,2,3,4,5,6,7,8\}$ are there?

$\binom{8}{4}$ is the answer.

How do we know we're done counting?
Why didn't we count the steps to the right?

We've chosen the 4 steps up. Of the remaining 4 steps, we'll choose 4 of them to be to the right.

$\binom{4}{4} = 1$. So multiplying by it doesn't change anything.

$$\frac{\cancel{4!}}{\cancel{4!}\,0!} = \frac{1}{0!} = 1$$

We're done counting when given the choices of our sequential process, we know exactly which path. And given a path we know exactly the choices of our sequential process.

Outline

So Far

Sum and Product Rules

Combinations and Permutations

Introduce ordering and remove it to make calculations easier

This Time

Some Proofs by counting two ways

Binomial Theorem

Principle of Inclusion-Exclusion

Overcounting

How many anagrams are there of SEATTLE $\rightarrow$ ESETALT

(an anagram is a rearrangement of letters).

It's not 7! That counts SEATTLE and SEATTLE as different things!

I swapped the Es (or maybe the Ts)

Overcounting

How many anagrams are there of SEATTLE

SEATTLE
SEATTLE
SEATTLE
SEATTLE

Pretend the order of the Es (and Ts) relative to each other matter (that SEATTLE and SEATTLE are different)

How many arrangements of SEATTLE? 7!

How have we overcounted? Es relative to each other and Ts relative to each other $2! \cdot 2!$

Final answer $\frac{7!}{2! \cdot 2!}$

Overcounting

How many anagrams are there of GODOGGY?

7 characters ~~tot~~ total

↳ 3 G's

↳ 2 O's

$$\frac{7!}{3!\,2!}$$

Overcounting

How many anagrams are there of GODOGGY?

$$\frac{7!}{2!3!}$$

One more piece of notation – "multinomial coefficient"

$\binom{7}{2,3}$ is alternate notation for $\frac{7!}{2!3!}$.

In general: $\binom{n}{k_1,k_2,\ldots,k_\ell} = \frac{n!}{k_1!\cdot k_2!\cdots k_\ell!}$

Popular notation among mathematicians.

Combination Facts

Some Facts about combinations

Symmetry of combinations: $\binom{n}{k} = \binom{n}{n-k}$

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Two Proofs of Symmetry

Proof 1: By algebra

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$ Definition of Combination

$$= \frac{n!}{(n-k)!k!}$$ Algebra (commutativity of multiplication)

$$= \binom{n}{n-k}$$ Definition of Combination

Two Proofs of Symmetry

Wasn’t that a great proof.

Airtight. No disputing it.

Got to say “commutativity of multiplication.”

But…do you know *why*? Can you *feel* why it’s true?

Two Proofs of Symmetry

"combinatorial proof"
"proof by double counting"

Suppose you have n people, and need to choose k people to be on your team. We will count the number of possible teams two different ways.

Way 1: We choose the k people to be on the team. Since order doesn't matter (you're on the team or not), there are $\binom{n}{k}$ possible teams.

Way 2: We choose the $n-k$ people to NOT be on the team. Everyone else is on it. Since order again doesn't matter, there are $\binom{n}{n-k}$ possible ways to choose the team.

Since we're counting the same thing, the numbers must be equal.
So $\binom{n}{k} = \binom{n}{n-k}$.

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-1-[k-1])!} + \frac{(n-1)!}{k!(n-1-k)!} && \text{definition of combination} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} && \text{subtraction} \\
&= \frac{[(n-1)!k!(n-k-1)!] + [(n-1)!(k-1)!(n-k)!]}{k!(k-1)!(n-k)!(n-k-1)!} && \text{Find a common denominator} \\
&= \frac{(n-1)!(k-1)!(n-k-1)!\,[k+(n-k)]}{k!(k-1)!(n-k)!(n-k-1)!} && \text{factor out common terms} \\
&= \frac{(n-1)!\,[k+(n-k)]}{k!(n-k)!} && \text{Cancel } (k-1)!(n-k-1)! \\
&= \frac{(n-1)!\cdot n}{k!(n-k)!} = \frac{n!}{k!(n-k)!} && \text{Algebra} \\
&= \binom{n}{k} && \text{Definition of combination}
\end{aligned}$$

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

You and $n-1$ other people are trying out for a k person team. How many possible teams are there?

Way 1: There are n people total, of which we're choosing k (and since it's a team order doesn't matter) $\binom{n}{k}$.

Way 2: There are two types of teams. Those for which you make the team, and those for which you don't.

If you do make the team, then $k-1$ of the other $n-1$ also make it.

If you don't make the team, k of the other $n-1$ also make it.

Overall, by sum rule, $\binom{n-1}{k-1} + \binom{n-1}{k}$.

Since we're computing the same number two different ways, they must be equal. So: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Takeaways

Formulas for factorial, permutations, combinations.

A useful trick for counting is to pretend order matters, then account for the overcounting at the end (by dividing out repetitions)

When trying to prove facts about counting, try to have each side of the equation count the same thing.

Much more fun and much more informative than just churning through algebra.

Binomial Theorem

In high school you probably memorized

$(x + y)^2 = x^2 + 2xy + y$^2

And $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

The Binomial Theorem tells us what happens for every n:

The Binomial Theorem

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Some intuition

$(x+y)(x+y)(x+y)\cdots(x+y)$

The Binomial Theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

$x^{?}y^{?}$

Intuition: Every monomial on the right-hand-side has either x or y from each of the terms on the left.

How many copies of $x^i y^{n-i}$ do you get? Well how many ways are there to choose i x's and $n-i$ y's? $\binom{n}{i}$.

Formal proof? Induction!

So What?

Well…if you saw it before, now you have a better understanding now of why it's true.

There are also a few cute applications of the binomial theorem to proving other theorems (usually by plugging in numbers for x and y) – you'll do one on HW1.

For example, set $x = 1$ and $y = 1$ then

$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i}$.

i.e. if you sum up binomial coefficients, you get 2^n. Exercise: reprove this equation (directly) with a combinatorial proof (where have we seen 2^n recently?)

Principle of Inclusion-Exclusion

Example

How many length 5 strings over the alphabet $\{a, b, c, \ldots, z\}$ contain:

Exactly 2 'a's OR

Exactly 1 'b' OR

No 'x's

For what A, B, C do we want $|A \cup B \cup C|$?

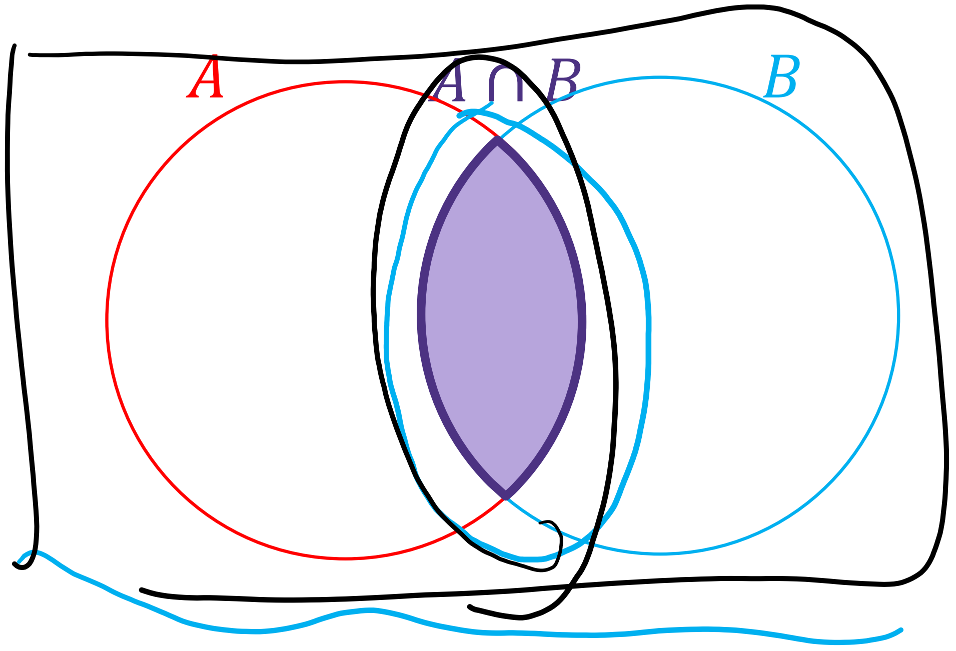
Principle of Inclusion-Exclusion

The sum rule says when A and B are disjoint (no intersection), then $|A \cup B| = |A| + |B|$.

What about when A and B aren't disjoint?

For two sets:

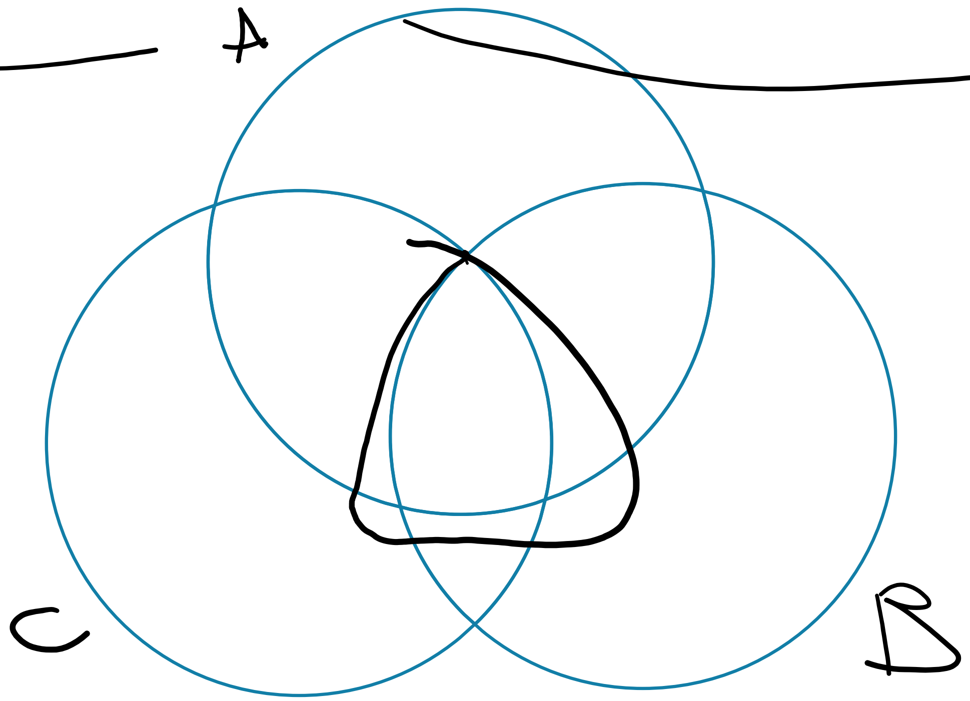
$$|A \cup B| = |A| + |B| - |A \cap B|$$



Principle of Inclusion-Exclusion

For three sets:

$$|A \cup B \cup C| = \boxed{|A|} + \boxed{|B|} + \boxed{|C|} - \boxed{|A \cap B|} - \boxed{|B \cap C|} - \boxed{|A \cap C|} + \boxed{|A \cap B \cap C|}$$



Example

How many length 5 strings over the alphabet $\{a, b, c, \ldots, z\}$ contain:

Exactly 2 ‘a’s OR

Exactly 1 ‘b’ OR

No ‘x’s

For what A, B, C do we want $|A \cup B \cup C|$?

In general:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| =$$

$$|A_1| + |A_2| + \cdots + |A_n|$$

$$-(|A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_1 \cap A_n| + |A_2 \cap A_3| + \cdots + |A_{n-1} \cap A_n|)$$

$$+(|A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|)$$

$$- \ \ldots$$

$$+(-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|$$

Add the individual sets, subtract all pairwise intersections, add all three-wise intersections, subtract all four-wise intersections,…, [add/subtract] the n-wise intersection.

Example

How many length 5 strings over the alphabet $\{a, b, c, \dots, z\}$ contain:
Exactly 2 'a's OR
Exactly 1 'b' OR
No 'x's

$|A \cup B \cup C|$

$A = \{$length 5 strings that contain exactly 2 'a's$\}$

$B = \{$length 5 strings that contain exactly 1 'b's$\}$

$C = \{$length 5 strings that contain no 'x's $\}$

$|A| = \binom{5}{2} \cdot 25^3$ (need to choose which "spots" are 'a' and remaining string)

$|B| = \binom{5}{1} \cdot 25^4$

$|C| = 25^5$

Example

How many length 5 strings over the alphabet $\{a, b, c, \dots, z\}$ contain:
Exactly 2 ‘a’s OR
Exactly 1 ‘b’ OR
No ‘x’s

$A = \{$length 5 strings that contain exactly 2 ‘a’s$\}$
$B = \{$length 5 strings that contain exactly 1 ‘b’s$\}$
$C = \{$length 5 strings that contain no ‘x’s $\}$

$|A| = \binom{5}{2} \cdot 25^3$
$|B| = \binom{5}{1} \cdot 25^{\cancel{3}\,4}$
$|C| = 25^5$

$|A \cap B| = \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2$ (choose ‘a’ spots, ‘b’ spot, remaining chars)
$|A \cap C| = \binom{5}{2} \cdot 24^3$ (choose ‘a’ spots, remaining [non-‘x’] chars)
$|B \cap C| = \binom{5}{1} \cdot 24^4$
$|A \cap B \cap C| = \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2$ (choose ‘a’ spots, ‘b’ spot, remaining [non-‘x’] chars)

Example

How many length 5 strings over the alphabet $\{a, b, c, \ldots, z\}$ contain:
Exactly 2 'a's OR
Exactly 1 'b' OR
No 'x's

$|A| = \binom{5}{2} \cdot 25^3$

$|B| = \binom{5}{1} \cdot 25^4$

$|C| = 25^5$

$|A \cap B| = \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2$

$|A \cap C| = \binom{5}{2} \cdot 24^3$

$|B \cap C| = \binom{5}{1} \cdot 24^4$

$|A \cap B \cap C| = \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2$

$$
\begin{aligned}
|A \cup B \cup C| &= \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= \binom{5}{2} \cdot 25^3 + \binom{5}{1} \cdot 25^4 + 25^5 - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= 11{,}875{,}000 - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= 11{,}875{,}000 - \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2 - \binom{5}{2} \cdot 24^3 - \binom{5}{1} \cdot 24^4 + |A \cap B \cap C| \\
&= 11{,}875{,}000 - 1{,}814{,}400 + |A \cap B \cap C| \\
&= 10{,}060{,}600 + |A \cap B \cap C| \\
&= 10{,}060{,}600 + \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2 \\
&= 10{,}060{,}600 + 15{,}870 \\
&= 10{,}076{,}470
\end{aligned}
$$

Practical tips

Give yourself clear definitions of A, B, C.

Make a table of all the formulas you need before you start actually calculating.

Calculate “size-by-size” and incorporate into the total.

Basic check: If (in an intermediate step) you ever:

1. Get a negative value
2. Get a value greater than the prior max by adding (after all the single sets)
3. Get a value less than the prior min by subtracting (after all the pairwise intersections)

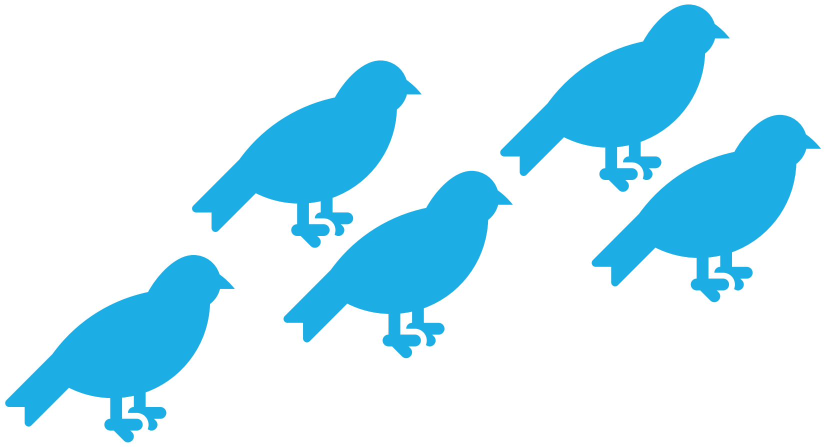
Then something has gone wrong.

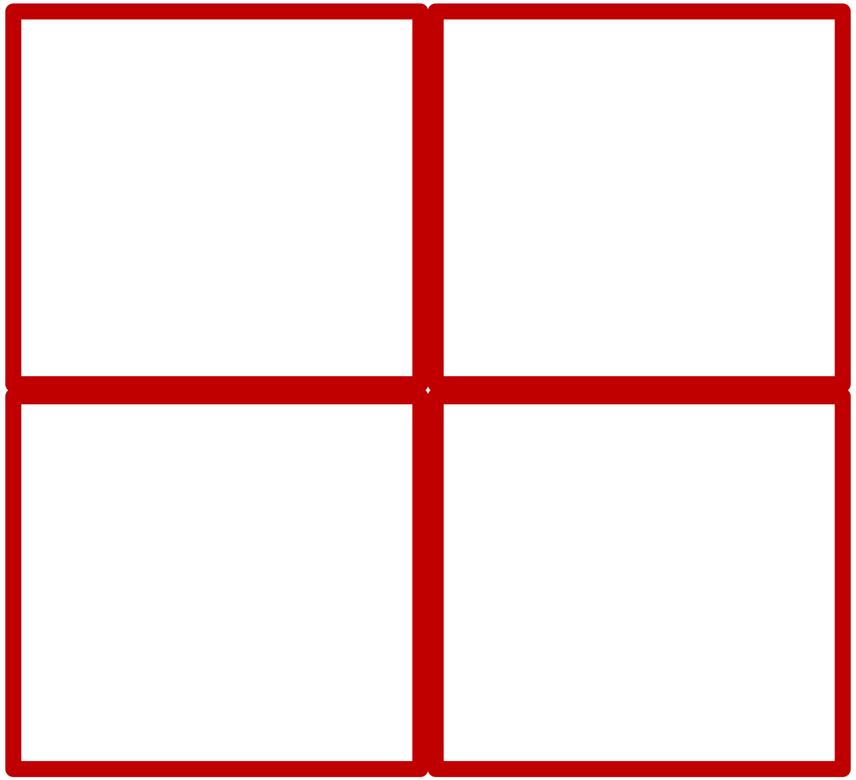
Pigeonhole Principle

Pigeonhole Principle

If you have 5 pigeons, and place them into 4 holes, then…

At least two pigeons are in the same hole.



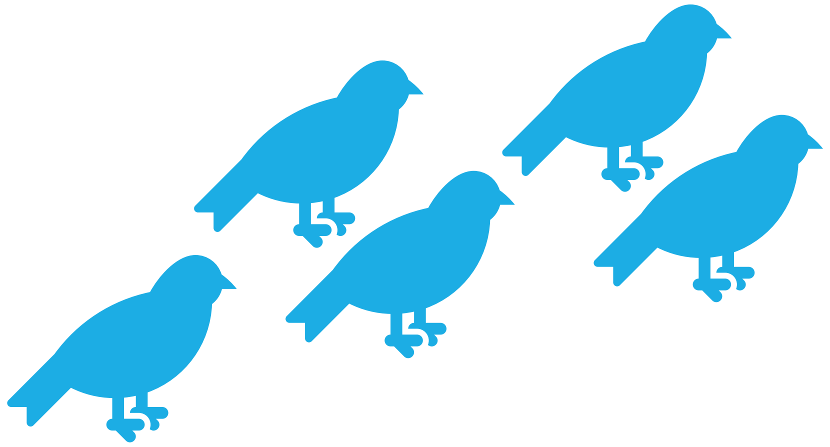


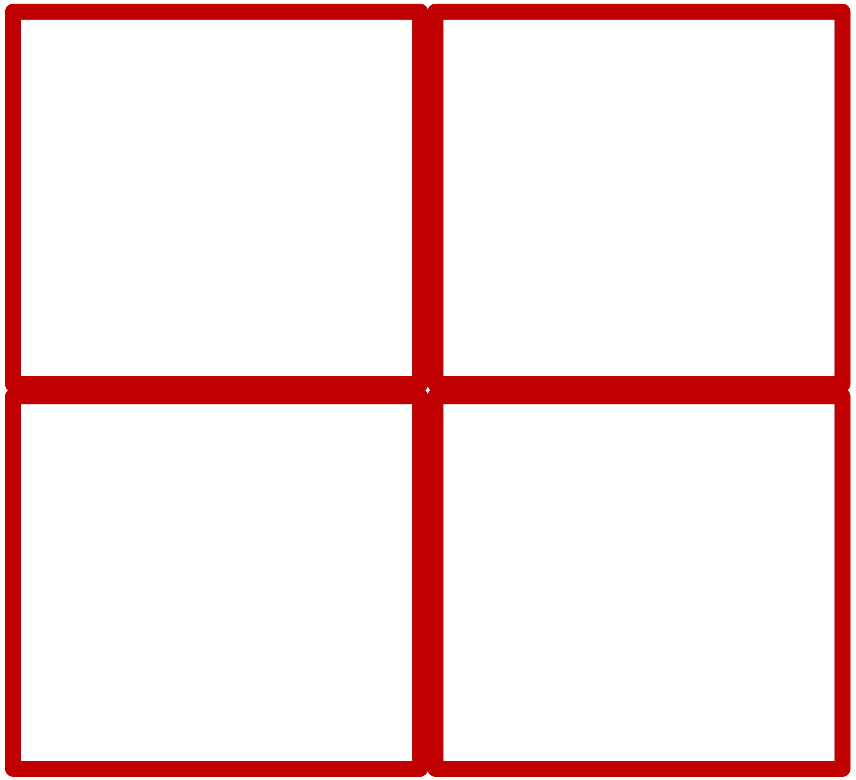
Pigeonhole Principle

If you have 5 pigeons, and place them into 4 holes, then…

At least two pigeons are in the same hole.

It might be more than two.





Strong Pigeonhole Principle

If you have n pigeons and k pigeonholes, then there is at least one pigeonhole that has at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons.

$\lceil a \rceil$ is the "ceiling" of a (it means always round up, $\lceil 1.1 \rceil = 2$, $\lceil 1 \rceil = 1$).

An example

If you have to take 10 classes, and have 3 quarters to take them in, then…

Pigeons: The classes to take

Pigeonholes: The quarter

Mapping: Which class you take the quarter in.

Applying the (generalized) pigeonhole principle, there is at least one quarter where you take at least $\left\lceil \frac{10}{3} \right\rceil = 4$ courses.

Practical Tips

When the pigeonhole principle is the right tool, it's usually the first thing you'd think of or the absolute last thing you'd think of.

For **really** tricky ones, we'll warn you in advance that it's the right method (you'll see one in the section handout).

When applying the principle, say:

- What are the pigeons
- What are the pigeonholes
- How do you map from pigeons to pigeonholes

Look for – a set you're trying to divide into groups, where collisions would help you somehow.