Probability

CSE 312 Spring 21
Lecture 5

Announcements

HW1 is due tonight (you can use late days just by [re]-submitting it later)

We made the deadline 11:59 PM, but please don't stay up all night if it interferes with your classes tomorrow.

HW2 will come out this evening, due next Wednesday.

Office hours have started! Robbie has hours right after A lecture.

(none today) but we have some office hours outside regular work hours (very early Tuesday morning, Tuesday night) every week.

Today

So far…we've done a lot of counting.

Starting today, we get to calculate probabilities!

Mostly notation and vocabulary today.

Probability

Probability is a way of **quantifying** our uncertainty.

When more than one outcome is possible,

To have “real-world” examples, we’ll need to start with some foundational processes that we’re going to assert exist

- We can flip a coin, and each face is equally likely to come up
- We can roll a die, and every number is equally likely to come up
- We can shuffle a deck of cards so that every ordering is equally likely.

Sample Space

Sample Space

A sample space Ω is the set of all possible outcomes of an experiment.

Examples:

For a single coin flip, $\Omega = \{H, T\}$

For a series of two coin flips, $\Omega = \{HH, HT, TH, TT\}$

For rolling a (normal) die: $\Omega = \{1,2,3,4,5,6\}$

Events

Event

An event $E \subseteq \Omega$ is a subset of possible outcomes (i.e. a subset of Ω)

Examples:

Get at least one head among two coin flips ($E = \{HH, HT, TH\}$)

Get an even number on a die-roll ($E = \{2,4,6\}$).

Examples

I roll a blue 4-sided die and a red 4-sided die.

The table contains the sample space.

The event "the sum of the dice is even" is in gold
The event "the blue die is 1" is in green

	D2=1	D2=2	D2=3	D2=4
D1=1	(1,1)	(1,2)	(1,3)	(1,4)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)

Probability

Probability

A probability is a number between 0 and 1 describing how likely a particular outcome is.

We'll define a function

$\mathbb{P}: \Omega \to [0,1]$

i.e. $\mathbb{P}$ takes an element of Ω as input and outputs the probability of the outcome.

We'll also use $\Pr[\omega]$, $P(\omega)$ as notation.

Example

Imagine we toss one coin.

Our sample space $\Omega = \{H, T\}$

What do you want $\mathbb{P}$ to be?

Example

Imagine we toss one coin.

Our sample space $\Omega = \{H, T\}$

What do you want $\mathbb{P}$ to be?

It depends on what we want to model

If the coin is fair $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$.

But we also might have a biased coin: $\mathbb{P}(H) = .85$, $\mathbb{P}(T) = 0.15$.

Probability Space

Probability Space

A (discrete) probability space is a pair $(\Omega, \mathbb{P})$ where:

Ω is the sample space

$\mathbb{P}: \Omega \to [0,1]$ is the probability measure.

$\mathbb{P}$ satisfies:

- $\mathbb{P}(x) \geq 0$ for all x
- $\sum_{x \in \Omega} \mathbb{P}(x) = 1$

- If $E, F \subseteq \Omega$ and $E \cap F = \emptyset$ then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

Probability Space

Flip a fair coin and roll a fair (6-sided) die.

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

Is this a valid probability space?

$\mathbb{P}$ takes in elements of Ω and outputs numbers between 0 and 1

$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$.

Measure

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

So what's the probability of seeing a heads?

Seeing heads isn't an element of the sample space!

Define $\mathbb{P}(E) = \sum_{\omega \in \mathrm{E}} \mathbb{P}(\omega)$

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Uniform Probability Space

The most common probability measure is the **uniform** probability measure. In the uniform measure, for every event E

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|}.$$

Let your sample space be all possible outcomes of a sequence of 100 coin tosses. Assign the uniform measure to this sample space. What is the probability of the event "there are exactly 50 heads?

A. $\binom{100}{50}/2^{100}$

B. $1/101$

C. $1/2$

D. $1/2^{50}$

E. There is not enough information in this problem.

Mutually Exclusive Events

Two events E, F are mutually exclusive if they cannot happen simultaneously.

In notation, $E \cap F = \emptyset$ (i.e. they're disjoint subsets of the sample space).

For example, if $\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$E_1 =$"the coin came up heads"

$E_2 =$"the coin came up tails"

$E_3 =$"the die showed an even number"

E_1 and E_2 are mutually exclusive.
E_1 and E_3 are not mutually exclusive.

Axioms and Consequences

We wrote down 3 requirements (axioms) on probability measures

- $\mathbb{P}(x) \geq 0$ for all x (non-negativity)
- $\sum_{x \in \Omega} \mathbb{P}(x) = 1$ (normalization)
- If E and F are mutually exclusive then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$ (countable additivity)

These lead quickly to these three corollaries

- $\mathbb{P}(\bar{E}) = 1 - \mathbb{P}(E)$ (complementation)
- If $E \subseteq F$, then $\mathbb{P}(E) \leq \mathbb{P}(F)$ (monotonicity)
- $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$ (inclusion-exclusion)

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What is your sample space?

What is your probability measure $\mathbb{P}$?

What is your event?

What is the probability?

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What is your sample space? $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

What is your probability measure $\mathbb{P}$? $\mathbb{P}(\omega) = 1/36$ for all $\omega \in \Omega$

What is your event? $\{2,4,6\} \times \{2,4,6\}$

What is the probability? $3^2/6^2$

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What if we can't tell the dice apart and always put the dice in increasing order by value.

What is your sample space?

$\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,3), (2,4), (2,5), (2,6)$

$(3,3), (3,4), (3,5), (3,6), (4,4), (4,5), (4,6), (5,5), (5,6), (6,6)\}$

What is your probability measure $\mathbb{P}$?

$\mathbb{P}((x, y)) = 2/36$ if $x \neq y$, $\mathbb{P}(x, x) = 1/36$

What is your event? $\{(2,2), (4,4), (6,6), (2,4), (2,6), (4,6)\}$

What is the probability? $3 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} = \frac{9}{36}$

Takeaways

There is often more than one sample space possible! But one is probably easier than the others.

Finding a sample space that will make the uniform measure correct will probably make finding the probabilities easier to calculate.

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space

Probability Measure

Event

Probability

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: $\{(x, y): x$ and y are different cards $\}$

Probability Measure uniform measure $\mathbb{P}(\omega) = \frac{1}{52 \cdot 51}$

Event: all pairs with equal values

Probability $\frac{13 \cdot P(4,2)}{52 \cdot 51}$

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: Set of all orderings of all 52 cards

Probability Measure uniform measure $\mathbb{P}(\omega) = \frac{1}{52!}$

Event: all lists that start with two cards of the same value

Probability $\frac{13 \cdot P(4,2) \cdot 50!}{52!}$

Takeaway

There's often information you "don't need" in your sample space.

It won't give you the wrong answer.

But it sometimes makes for extra work/a harder counting problem,

Good indication: you cancelled A LOT of stuff that was common in the numerator and denominator.

Balls and Urns

You have an urn* with two red balls and two green balls inside.

Take out two of the balls replacing the first ball after you take it out.

What's the probability of drawing out both red balls?

Sequential process: $\frac{1}{2}$ probability of the first being red

$\frac{1}{2}$ probability of the second being red.

An urn is a vase