

Variance

CSE 312 Spring 21
Lecture 13

Announcements

Two typo fixes in the homework.

Problem 1, the example lost a negative sign

Problem 6, the example's second bullet assigned one test to two people (that can't happen). It's corrected on the webpage as well.

Office hour shifts this week:

Robbie will stay after B lecture longer today (until about 3)

Howard's OH today are cancelled; moved to Wednesday 10:30

Where are we?

A random variable is a way to summarize what outcome you saw.

The Expectation of a random variable is its average value.

A way to summarize a random variable

Expectation is **linear**

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

$X + Y$ is a random variable – it's a function that outputs a number given an outcome (or, here, a combination of outcomes).

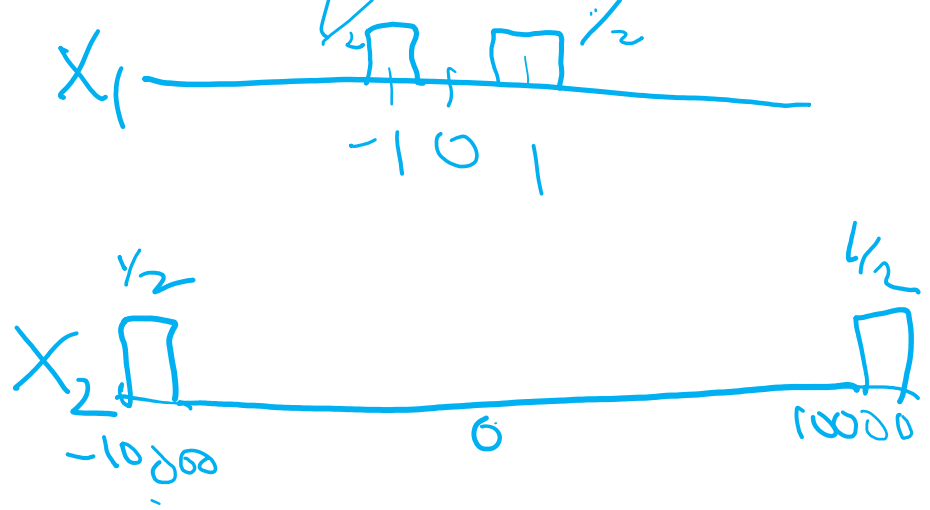
Variance

Another one number summary of a random variable.

But wait, we already have expectation, what's this for?

Consider these two games

Would you be willing to play these games?



Game 1: I will flip a fair coin; if it's heads, I pay you \$1. If it's tails, you pay me \$1. Let X_1 be your profit if you play game 1

Game 2: I will flip a fair coin; if it's heads, I pay you \$10,000. If it's tails, you pay me \$10,000. Let X_2 be your profit if you play game 2.

Both games are "fair" ($\mathbb{E}[X_1] = \mathbb{E}[X_2] = 0$)

What's the difference

Expectation tells you what the average will be...

But it doesn't tell you how "extreme" your results could be.

Nor how likely those extreme results are.

Game 2 has many (well, only) very extreme results.

In expectation they "cancel out" but if you can only play once...

...it would be nice to measure that.

Designing a Measure – Try 1

Well let's measure how far all the events are away from the center, and how likely they are

$$\sum_{\omega} \mathbb{P}(\omega) \cdot (X(\omega) - \mathbb{E}[X])$$

What happens with Game 1?

$$\frac{1}{2} \cdot (1 - 0) + \frac{1}{2} \cdot (-1 - 0)$$
$$\frac{1}{2} - \frac{1}{2} = 0$$

What happens with Game 2?

$$\frac{1}{2} \cdot (100000 - 0) + \frac{1}{2} \cdot (-100000 - 0)$$
$$50000 - 50000 = 0$$

Designing a Measure – Try 2

How do we prevent cancelling? Squaring makes everything positive.

$$\sum_{\omega} \mathbb{P}(\omega) \cdot (X(\omega) - \mathbb{E}[X])^2$$

What happens with Game 1?

$$\frac{1}{2} \cdot (1 - 0)^2 + \frac{1}{2} \cdot (-1 - 0)^2$$
$$\frac{1}{2} + \frac{1}{2} = 1$$

What happens with Game 2?

$$\frac{1}{2} \cdot (100000 - 0)^2 + \frac{1}{2} \cdot (-100000 - 0)^2$$
$$5,000,000,000 + 5,000,000,000 = 10^{10}$$

Why Squaring

Why not absolute value? Or Fourth power?

Squaring is nicer algebraically.

Our goal with variance was to talk about the spread of results. Squaring makes extreme results even more extreme.

Fourth power over-emphasizes the extreme results (for our purposes).

Variance

Variance

The variance of a random variable X is

$$\text{Var}(X) = \sum_{\omega} \mathbb{P}(\omega) \cdot (X(\omega) - \mathbb{E}[X])^2 = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

The first form forms are the definition. The last one is an algebra trick.

Variance of a die

Let X be the result of rolling a fair die.

$$\begin{aligned}\text{Var}(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[(X - 3.5)^2] \\ &= \frac{1}{6}(1 - 3.5)^2 + \frac{1}{6}(2 - 3.5)^2 + \frac{1}{6}(3 - 3.5)^2 + \frac{1}{6}(4 - 3.5)^2 + \frac{1}{6}(5 - 3.5)^2 + \frac{1}{6}(6 - 3.5)^2 \\ &= \frac{35}{12} \approx 2.92.\end{aligned}$$

$$\text{Or } \mathbb{E}[X^2] - (E[X])^2 = \sum_{k=1}^6 \frac{1}{6} \cdot k^2 - 3.5^2 = \frac{91}{6} - 3.5^2 \approx 2.92$$

Variance of n Coin Flips

Flip a coin n times, where it comes up heads with probability p each time (independently). Let X be the total number of heads.

We saw last time $\mathbb{E}[X] = np$.

$$X_i = \begin{cases} 1 & \text{if flip } i \text{ is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbb{E}[X_i] = \sum_{i=1}^n p = np.$$

Variance of n Coin Flips

Flip a coin n times, where it comes up heads with probability p each time (independently). Let X be the total number of heads.

What about $\text{Var}(X)$

$$\begin{aligned}\mathbb{E}[(X - \mathbb{E}[X])^2] &= \sum_{\omega} \mathbb{P}(\omega)(X(\omega) - np)^2 \\ &= \sum_{k=0}^n \binom{n}{k} \cdot p^k (1-p)^{n-k} \cdot (k - np)^2\end{aligned}$$

Algebra time?

Variance

If X and Y are independent then
 $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$

We'll talk about what it means for random variables to be independent in a second...

For now, in this problem X_i is independent of X_j for $i \neq j$ where

$$X_i = \begin{cases} 1 & \text{if flip } i \text{ was heads} \\ 0 & \text{otherwise} \end{cases}$$

Variance

$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{Var}(X_i)$$

What's the $\text{Var}(X_i)$?

$$\mathbb{E}[(X_i - \mathbb{E}[X_i])^2]$$

$$= \mathbb{E}[(X_i - p)^2]$$

$$= p(1-p)^2 + (1-p)(0-p)^2$$

$$= p(1-p)[(1-p) + p] = p(1-p).$$

$$\text{OR } \text{Var}(X_i) = \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 = \mathbb{E}[X_i] - p^2 = p - p^2 = p(1-p).$$

Plugging In

$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{Var}(X_i)$$

What's the $\text{Var}(X_i)$?

$$p(1 - p).$$

$$\text{Var}(X) = \sum_{i=1}^n p(1 - p) = np(1 - p).$$

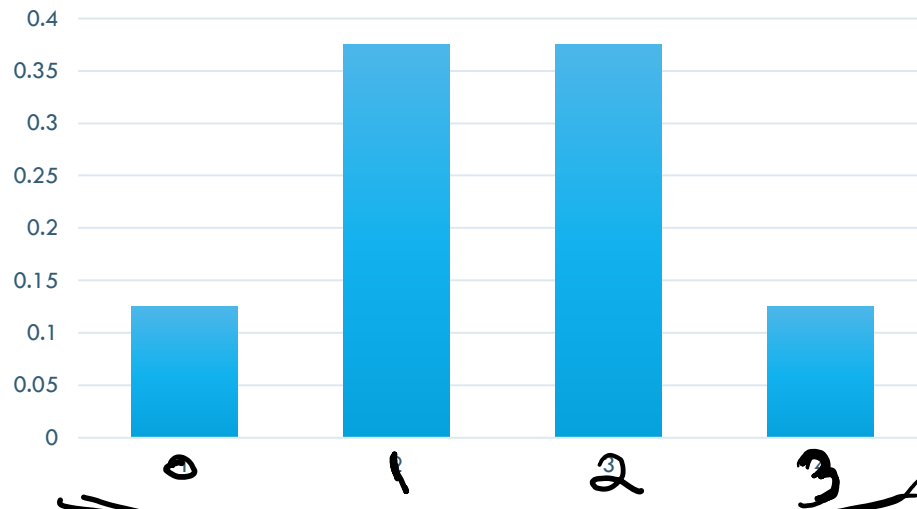
Expectation and Variance aren't everything

Alright, so expectation and variance is everything right?

No!

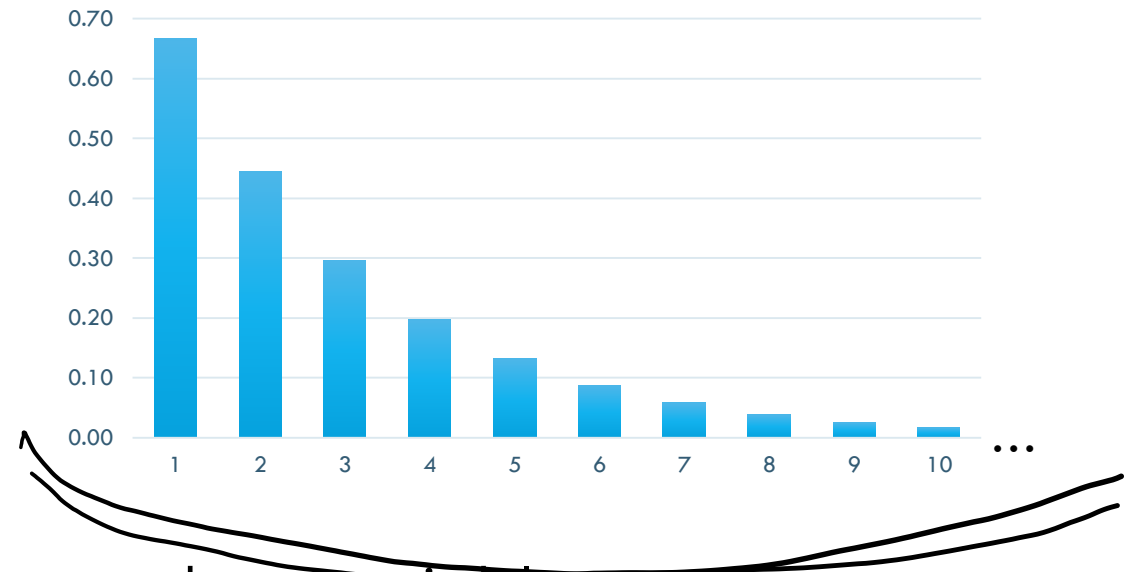
Flip a fair coin 3 times indep. Count heads.

PMF 1 with $E=3/2$, $Var=3/4$



Flip a biased coin (prob heads=2/3) until heads. Count flips.

PMF 2 with $E=3/2$, $Var=3/4$

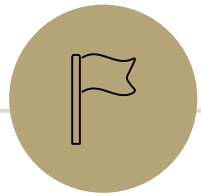


A PMF or CDF *does* fully describe a random variable.

Calculation Trick

$$\begin{aligned}\mathbb{E}[(X - \mathbb{E}[X])^2] &= \mathbb{E}[X^2 - 2X\mathbb{E}[X] + (\mathbb{E}[X])^2] \text{ expanding the square} \\ &= \mathbb{E}[X^2] - \mathbb{E}[2X\mathbb{E}[X]] + \mathbb{E}[(\mathbb{E}[X])^2] \text{ linearity of expectation.} \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[(\mathbb{E}[X])^2] \text{ linearity of expectation.} \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + (\mathbb{E}[X])^2 \text{ expectation of a constant is the constant} \\ &= \mathbb{E}[X^2] - 2(\mathbb{E}[X])^2 + (\mathbb{E}[X])^2 \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2\end{aligned}$$

$$\text{So } \text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$



Independence

Independence of Random Variables

That's for events...what about random variables?

Independence (of random variables)

X and Y are independent if for all k, ℓ

$$\mathbb{P}(X = k, Y = \ell) = \mathbb{P}(X = k)\mathbb{P}(Y = \ell)$$

We'll often use commas instead of \cap symbol.

Independence of Random Variables

The “for all values” is important.

We say that the event “the sum is 7” is independent of “the red die is 5”
What about S = “the sum of two dice” and R = “the value of the red die”

Independence of Random Variables

The “for all values” is important.

We say that the event “the sum is 7” is independent of “the red die is 5”

What about S = “the sum of two dice” and R = “the value of the red die”

NOT independent.

$$\mathbb{P}(S = 2, R = 5) \neq \mathbb{P}(S = 2)\mathbb{P}(R = 5) \text{ (for example)}$$

Independence of Random Variables

Flip a coin independently $2n$ times.

Let X be "the number of heads in the first n flips."

Let Y be "the number of heads in the last n flips."

X and Y are independent.

Mutual Independence for RVs

A little simpler to write down than for events

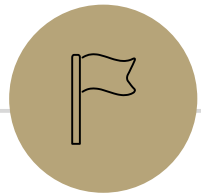
Mutual Independence (of random variables)

X_1, X_2, \dots, X_n are mutually independent if for all x_1, x_2, \dots, x_n

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = \mathbb{P}(X_1 = x_1)\mathbb{P}(X_2 = x_2) \cdots \mathbb{P}(X_n = x_n)$$

DON'T need to check all subsets for random variables...

But you do need to check all values (all possible x_i) still.



Common Distributions



What Does Independence Give Us?

If X and Y are independent random variables, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

$$\mathbb{E}[X \cdot Y] = \mathbb{E}[X]\mathbb{E}[Y]$$

$$\mathbb{E}\left[\frac{X}{Y}\right] = \mathbb{E}[X] \cdot \mathbb{E}\left[\frac{1}{Y}\right]$$

Facts About Variance

$$\text{Var}(X + c) = \text{Var}(X)$$

Proof:

$$\begin{aligned}\text{Var}(X + c) &= \mathbb{E}[(X + c)^2] - \mathbb{E}[X + c]^2 \\ &= \mathbb{E}[X^2] + \mathbb{E}[2Xc] + \mathbb{E}[c^2] - (\mathbb{E}[X] + c)^2 \\ &= \mathbb{E}[X^2] + 2c\mathbb{E}[X] + c^2 - \mathbb{E}[X]^2 - 2c\mathbb{E}[X] - c^2 \\ &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ &= \text{Var}(X)\end{aligned}$$

Facts about Variance

$$\begin{aligned}\text{Var}(aX) &= a^2 \text{Var}(X) \\ &= \mathbb{E}[(aX)^2] - (\mathbb{E}[aX])^2 \\ &= a^2 \mathbb{E}[X^2] - (a\mathbb{E}[X])^2 \\ &= a^2 \mathbb{E}[X^2] - a^2 \mathbb{E}[X]^2 \\ &= a^2 (\mathbb{E}[X^2] - \mathbb{E}[X]^2)\end{aligned}$$