No activity slide today

Continuous Zoo

Announcements

See the Lecture 16 FAQ for some typo corrections from Monday.

HW6 will come out tonight.

A little longer than HW5 was, but hopefully still a bit shorter than HW4.

One programming question on HW6.

Announcements

Friday's lecture will be unusual – it'll be "asynchron-ish"

Here's the plan:

Robbie will record the lecture and release it to panopto Wednesday night or Thursday morning.

It'll cover whatever is left from this slide deck, and an introduction to the programming project that's on HW6.

If you want to get started on that project right away, watch lecture early!

Friday, both lecture's zooms will be office hours.
Robbie will be there if vaccine side-effects are ok, otherwise a TA will fill in.

Continuous RVs

$$F_X(k) = \mathbb{P}(X \leq k) = \int_{-\infty}^{k} f_X(z)\,dz$$

$$\mathbb{P}(a \leq X \leq b) = \int_{a}^{b} f_X(z)\,dz = F_X(b) - F_X(a)$$

$$\mathbb{P}(X = k) = \int_{k}^{k} f_X(z)\,dz = 0 \text{ (for any constant } k)$$

Densities are not normalized to be between 0 and 1. Write out the pdf for a random real number between 0 and ½ to confirm this fact.

CDF is increasing, $\lim_{k \to -\infty} f_X(k) = 0$; $\lim_{k \to \infty} f_X(k) = 1$

Let's calculate an expectation

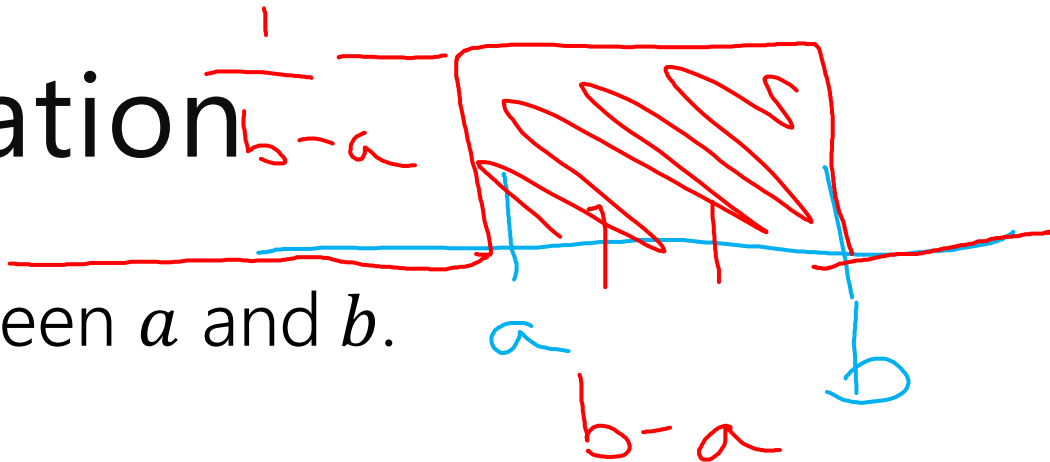Let X be a uniform random number between a and b.

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} z \cdot f_X(z) \, dz$$

$$= \int_{-\infty}^{a} z \cdot 0 \, dz + \int_{a}^{b} z \cdot \frac{1}{b-a} \, dz + \int_{b}^{\infty} z \cdot 0 \, dz$$

$$= 0 + \int_{a}^{b} \frac{z}{b-a} \, dz + 0$$

$$= \frac{z^2}{2(b-a)} \Big|_{z=a}^{b} = \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} = \frac{b^2 - a^2}{2(b-a)} = \frac{(b+a)(b-a)}{2(b-a)} = \frac{a+b}{2}$$

What about $\mathbb{E}[g(X)]$

Let $X \sim \text{Unif}(a,b)$, what about $\mathbb{E}[X^2]$?

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} z^2 f_X(z) \, dz$$

$$= \int_{-\infty}^{a} z^2 \cdot 0 \, dz + \int_{a}^{b} z^2 \cdot \frac{1}{b-a} \, dz + \int_{b}^{\infty} z^2 \cdot 0 \, dz$$

$$= 0 + \int_{a}^{b} z^2 \cdot \frac{1}{b-a} \, dz + 0$$

$$= \frac{1}{b-a} \cdot \frac{z^3}{3} \bigg|_{z=a}^{b} = \frac{1}{b-a} \left(\frac{b^3}{3} - \frac{a^3}{3} \right) = \frac{1}{3(b-a)} \cdot (b-a)(a^2 + ab + b^2)$$

$$= \frac{a^2 + ab + b^2}{3}$$

$$f_X(z) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq z \leq b \\ 0 & \text{o/w} \end{cases}$$

Let's assemble the variance

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{4(a^2+ab+b^2)}{12} - \frac{3(a^2+2ab+b^2)}{12}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$= \frac{(a-b)^2}{12}$$

$$= \frac{(b-a)^2}{12}$$

Continuous Uniform Distribution

$X \sim \text{Unif}(a, b)$ (uniform real number between a and b)

PDF: $f_X(k) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \leq k \leq b \\ 0 & \text{otherwise} \end{cases}$

CDF: $F_X(k) = \begin{cases} 0 & \text{if } k < a \\ \dfrac{k-a}{b-a} & \text{if } a \leq k \leq b \\ 1 & \text{if } k \geq b \end{cases}$

$\mathbb{E}[X] = \dfrac{a+b}{2}$

$\text{Var}(X) = \dfrac{(b-a)^2}{12}$

$$P(X = k) = \int_k^k f_X(z)\,dz = 0$$

$$\frac{\varepsilon \cdot f_X(k)}{\varepsilon \cdot f_X(\ell)} = \frac{P(X \approx k)}{P(X \approx \ell)}$$

Continuous Zoo

$X \sim \text{Unif}(a, b)$
$f_X(k) = \dfrac{1}{b-a}$
$\mathbb{E}[X] = \dfrac{a+b}{2}$
$\text{Var}(X) = \dfrac{(b-a)^2}{12}$

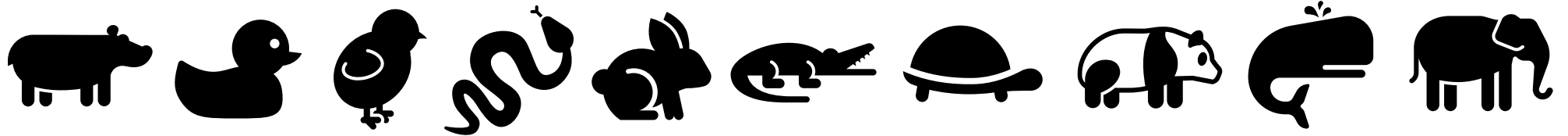| $X \sim \text{Exp}(\lambda)$ |
|:---:|
| $f_X(k) = \lambda e^{-\lambda k}$ for $k \geq 0$ |
| $\mathbb{E}[X] = \dfrac{1}{\lambda}$ |
| $\text{Var}(X) = \dfrac{1}{\lambda^2}$ |

$X \sim \mathcal{N}(\mu, \sigma^2)$
$f_X(k) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right)$
$\mathbb{E}[X] = \mu$
$\text{Var}(X) = \sigma^2$

It's a smaller zoo, but it's just as much fun!

Zoo!

$X \sim \mathrm{Unif}(a, b)$	$X \sim \mathrm{Ber}(p)$	$X \sim \mathrm{Bin}(n, p)$	$X \sim \mathrm{Geo}(p)$
$f_X(k) = \dfrac{1}{b - a + 1}$ $$\mathbb{E}[X] = \dfrac{a + b}{2}$$ $$\mathrm{Var}(X) = \dfrac{(b - a)(b - a + 2)}{12}$$	$f_X(0) = 1 - p; \ f_X(1) = p$ $$\mathbb{E}[X] = p$$ $$\mathrm{Var}(X) = p(1 - p)$$	$f_X(k) = \dbinom{n}{k} p^k (1 - p)^{n-k}$ $$\mathbb{E}[X] = np$$ $$\mathrm{Var}(X) = np(1 - p)$$	$f_X(k) = (1 - p)^{k-1} p$ $$\mathbb{E}[X] = \dfrac{1}{p}$$ $$\mathrm{Var}(X) = \dfrac{1 - p}{p^2}$$

$X \sim \mathrm{NegBin}(r, p)$	$X \sim \mathrm{HypGeo}(N, K, n)$	$X \sim \mathrm{Poi}(\lambda)$
$f_X(k) = \dbinom{k - 1}{r - 1} p^r (1 - p)^{k-r}$ $$\mathbb{E}[X] = \dfrac{r}{p}$$ $$\mathrm{Var}(X) = \dfrac{r(1 - p)}{p^2}$$	$f_X(k) = \dfrac{\dbinom{K}{k}\dbinom{N-K}{n-k}}{\dbinom{N}{n}}$ $$\mathbb{E}[X] = n\dfrac{K}{N}$$ $$\mathrm{Var}(X) = \dfrac{K(N - K)(N - n)}{N^2(N - 1)}$$	$f_X(k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$ $$\mathbb{E}[X] = \lambda$$ $$\mathrm{Var}(X) = \lambda$$

Exponential Random Variable

Like a geometric random variable, but continuous time. How long do we wait until an event happens? (instead of "how many flips until a heads")

Where waiting doesn't make the event happen any sooner.

Geometric: $\mathbb{P}(X = k + 1 \mid X \geq 1) = \mathbb{P}(X = k)$

When the first flip is tails, the coin doesn't remember it came up tails, you've made no progress.

For an exponential random variable:

$\mathbb{P}(Y \geq k + 1 \mid Y \geq 1) = \mathbb{P}(Y \geq k)$

Exponential random variable

If you take a Poisson random variable and ask "what's the time until the next event" you get an exponential distribution!

Let's find the CDF for an exponential.

Let $Y \sim \text{Exp}(\lambda)$, be the time until the first event, when we see an average of λ events per time unit. What's $\mathbb{P}(Y > t)$?

What Poisson are we waiting on? For $X \sim \text{Poi}(\lambda t)$ $\mathbb{P}(Y > t) = \mathbb{P}(X = 0)$

$$\mathbb{P}(X = 0) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$$

$$F_Y(t) = \mathbb{P}(Y \leq t) = 1 - e^{-\lambda t} \text{ (for } t \geq 0, \ F_Y(x) = 0 \text{ for } x < 0)$$

Find the density

We know the CDF, $F_Y(t) = \mathbb{P}(Y \le t) = 1 - e^{-\lambda t}$ $t \ge 0$

What's the density?

$f_Y(t) =$

Find the density

We know the CDF, $F_Y(t) = \mathbb{P}(Y \le t) = 1 - e^{-\lambda t}$
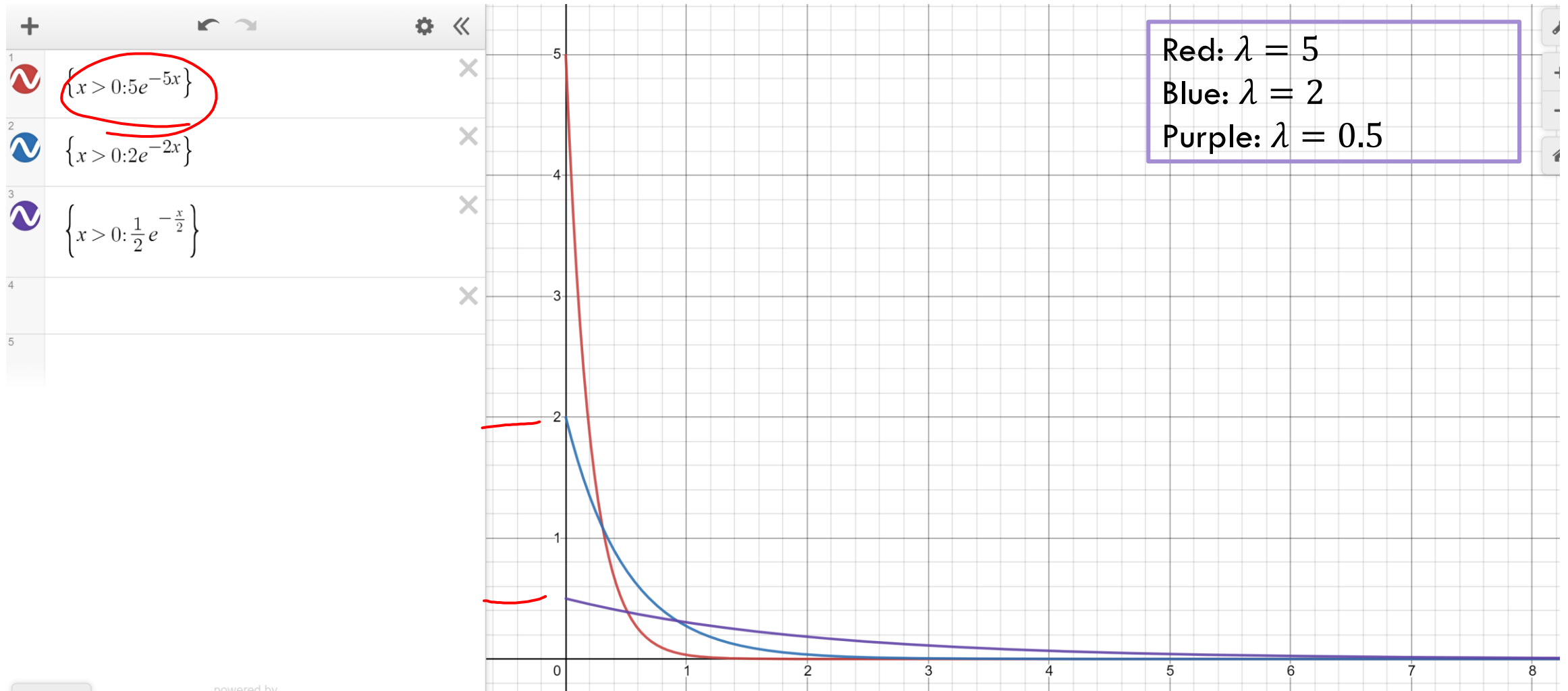
What's the density?

$$f_Y(t) = \frac{d}{dt}\left(1 - e^{-\lambda t}\right) = 0 - \frac{d}{dt}\left(e^{-\lambda t}\right) = \lambda e^{-\lambda t}.$$

For t $\ge$ 0 it's that expression

For $t < 0$ it's just **0.**

Exponential PDF

Red: $\lambda = 5$
Blue: $\lambda = 2$
Purple: $\lambda = 0.5$

Memorylessness

$$\mathbb{P}(X \geq k + 1 | X \geq 1) = \frac{\mathbb{P}(X \geq k+1 \cap X \geq 1)}{\mathbb{P}(X \geq 1)} = \frac{\mathbb{P}(X \geq k+1)}{1 - (1 - e^{-\lambda \cdot 1})}$$

$$= \frac{e^{-\lambda(k+1)}}{e^{-\lambda}} = e^{-\lambda k}$$

What about $\mathbb{P}(X \geq k)$ (without conditioning on the first step)?

$$1 - (1 - e^{-\lambda k}) = e^{-\lambda k}$$

It's the same!!!

More generally, for an exponential rv X, $\mathbb{P}(X \geq s + t | X \geq s) = \mathbb{P}(X \geq t)$

Side note

I hid a trick in that algebra,

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X < 1) = 1 - \mathbb{P}(X \leq 1)$$

The first step is the complementary law.

The second step is using that $\int_1^1 f_X(z)\mathrm{d}z = 0$

In general, for continuous random variables we can switch out $\leq$ and $<$ without anything changing.

We can't make those switches for discrete random variables.

Expectation of an exponential

Let $X \sim \text{Exp}(\lambda)$

$\mathbb{E}[X] = \int_{-\infty}^{\infty} z \cdot f_X(z) \mathrm{d}z$

$= \int_0^{\infty} z \cdot \lambda e^{-\lambda z} \, dz$

Let $u = z$; $dv = \lambda e^{-\lambda z} dz$ $(v = -e^{-\lambda z})$

Integrate by parts: $-ze^{-\lambda z} - \int -e^{-\lambda z} \, dz = -ze^{-\lambda z} - \frac{1}{\lambda} e^{-\lambda z}$

Definite Integral: $-ze^{-\lambda z} - \frac{1}{\lambda} e^{-\lambda z} \big|_{z=0}^{\infty} = \left(\lim_{z \to \infty} -ze^{-\lambda z} - \frac{1}{\lambda} e^{-\lambda z} \right) - \left(0 - \frac{1}{\lambda} \right)$

By L'Hopital's Rule $\left(\lim_{z \to \infty} -\frac{z}{e^{\lambda z}} - \frac{1}{\lambda e^{\lambda z}} \right) - \left(0 - \frac{1}{\lambda} \right) = \left(\lim_{z \to \infty} -\frac{1}{\lambda e^{\lambda z}} \right) + \frac{1}{\lambda} = \frac{1}{\lambda}$

Don't worry about the derivation (it's here if you're interested; you're not responsible for the derivation. Just the value.

Variance of an exponential

If $X \sim \text{Exp}(\lambda)$ then $\text{Var}(X) = \dfrac{1}{\lambda^2}$

Similar calculus tricks will get you there.

Exponential

$X \sim \text{Exp}(\lambda)$

Parameter $\lambda \geq 0$ is the average number of events in a unit of time.

$$f_X(k) = \begin{cases} \lambda e^{-\lambda k} & \text{if } k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(k) = \begin{cases} 1 - e^{-\lambda k} & \text{if } k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

$$\text{Var}(X) = \frac{1}{\lambda^2}$$

Normal Random Variable

X is a normal (aka Gaussian) random variable with mean μ and variance σ^2 (written $X \sim \mathcal{N}(\mu, \sigma^2)$) if it has the density:
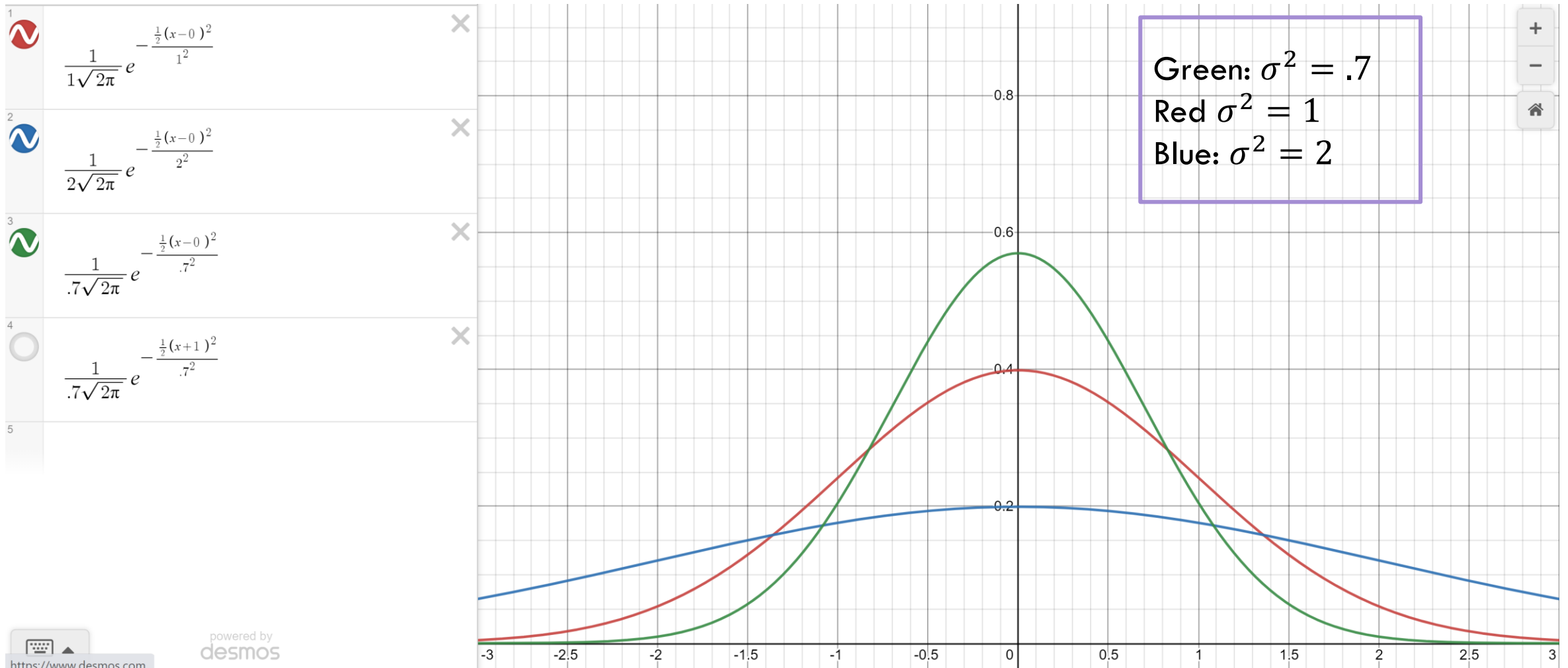
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

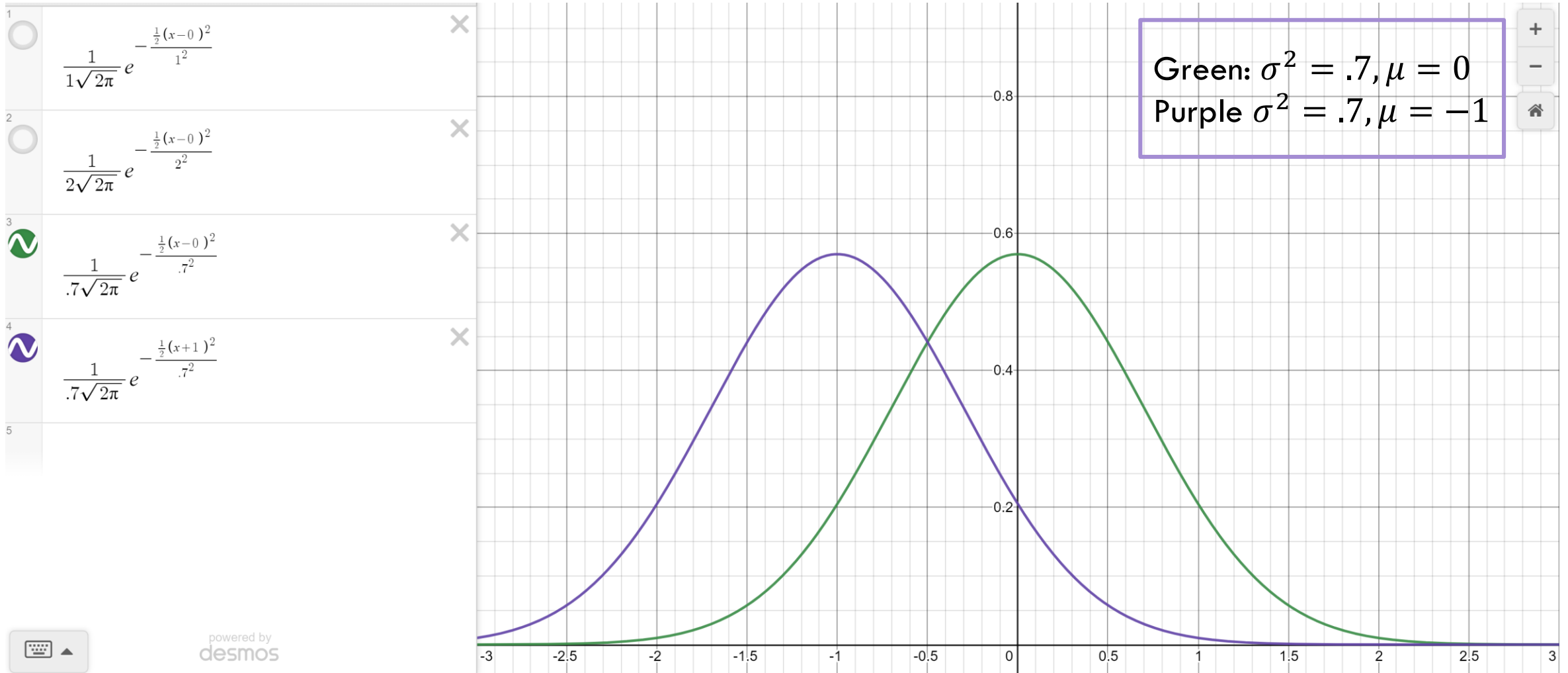Let's get some intuition for that density...

Is $\mathbb{E}[X] = \mu$?

Yes! Plug in $\mu - k$ and $\mu + k$ and you'll get the same density for every k. The density is symmetric around μ. The expectation must be μ.

Changing the variance

$$\frac{1}{1\sqrt{2\pi}} e^{-\frac{\frac{1}{2}(x-0)^2}{1^2}}$$

$$\frac{1}{2\sqrt{2\pi}} e^{-\frac{\frac{1}{2}(x-0)^2}{2^2}}$$

$$\frac{1}{.7\sqrt{2\pi}} e^{-\frac{\frac{1}{2}(x-0)^2}{.7^2}}$$

$$\frac{1}{.7\sqrt{2\pi}} e^{-\frac{\frac{1}{2}(x+1)^2}{.7^2}}$$

Green: $\sigma^2 = .7$
Red $\sigma^2 = 1$
Blue: $\sigma^2 = 2$

Changing the mean

$$\frac{1}{1\sqrt{2\pi}}e^{-\frac{\frac{1}{2}(x-0)^2}{1^2}}$$

$$\frac{1}{2\sqrt{2\pi}}e^{-\frac{\frac{1}{2}(x-0)^2}{2^2}}$$

$$\frac{1}{.7\sqrt{2\pi}}e^{-\frac{\frac{1}{2}(x-0)^2}{.7^2}}$$

$$\frac{1}{.7\sqrt{2\pi}}e^{-\frac{\frac{1}{2}(x+1)^2}{.7^2}}$$

Green: $\sigma^2 = .7, \mu = 0$
Purple $\sigma^2 = .7, \mu = -1$

Scaling Normals

When we scale a normal (multiplying by a constant or adding a constant) we get a normal random variable back!

If $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$

Then for $Y = aX + b$, $Y \sim \mathcal{N}\left(a\mu + b, a^2\sigma^2\right)$

Normals are unique in that you get a NORMAL back.

If you multiply a binomial by **3/2** you don't get a binomial (it's support isn't even integers!)

Normalize

To turn $X \sim \mathcal{N}(\mu, \sigma^2)$ into $Y \sim \mathcal{N}(0,1)$ you want to set

$$Y = \frac{X - \mu}{\sigma}$$

Why normalize?

The density is a mess. The CDF does not have a pretty closed form.
But we're going to need the CDF a lot, so…

Table of Standard Normal CDF

The way we'll evaluate the CDF of a normal is to:
1. convert to a standard normal
2. Round the "z-score" to the hundredths place.
3. Look up the value in the table.

It's 2021, we're using a table?

The table makes sure we have consistent rounding rules (makes it easier for us to debug with you).
You can't evaluate this by hand – the "z-score" can give you intuition right away.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Use the table!

We'll use the notation $\Phi(z)$ to mean $F_X(z)$ where $X \sim \mathcal{N}(0,1)$.

Let $Y \sim \mathcal{N}(5,4)$ what is $\mathbb{P}(Y > 9)$?

$\mathbb{P}(Y > 9)$

$= \mathbb{P}\left(\frac{Y-5}{2} > \frac{9-5}{2}\right)$ we've just written the inequality in a weird way.

$= \mathbb{P}(X > \frac{9-5}{2})$ where X is $\mathcal{N}(0,1)$.

$= 1 - \mathbb{P}\left(X \leq \frac{9-5}{2}\right) = 1 - \Phi(2.00) = 1 - 0.97725 = .02275.$