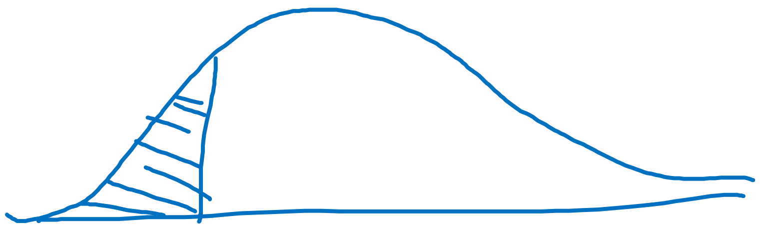


Normal Random Variables

CSE 312 Spring 21
Lecture 18

Normal Random Variable

X is a normal (aka Gaussian) random variable with mean μ and variance σ^2 (written $X \sim \mathcal{N}(\mu, \sigma^2)$) if it has the density:

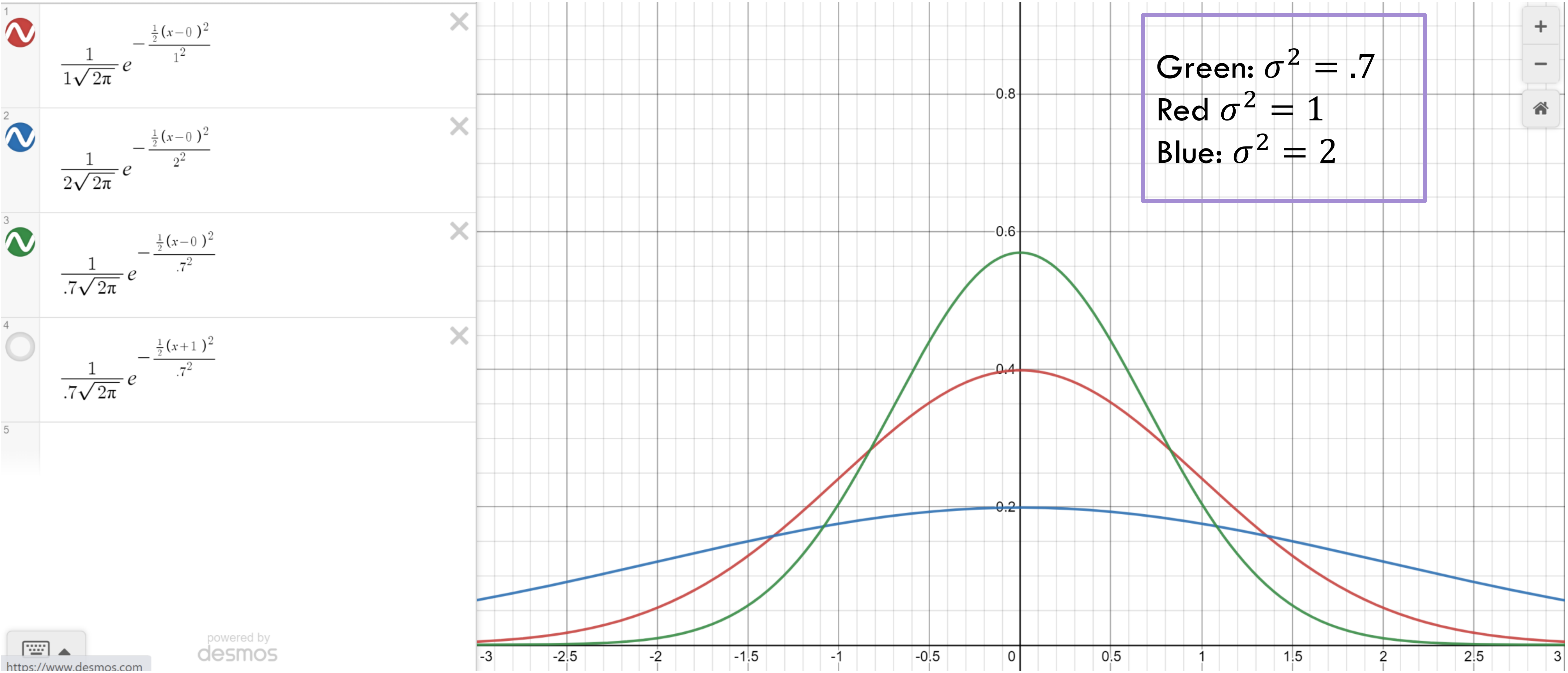
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Let's get some intuition for that density...

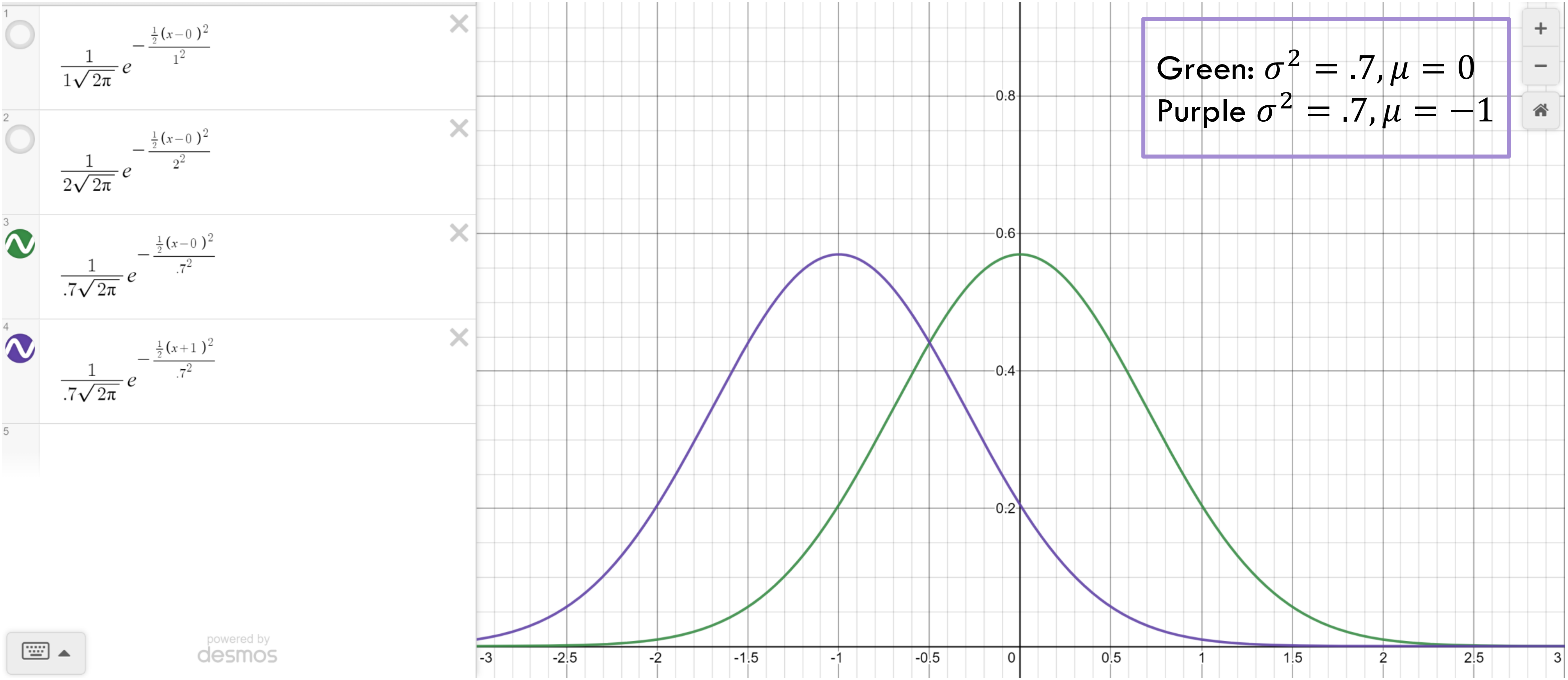
Is $\mathbb{E}[X] = \mu$?

Yes! Plug in $\mu - k$ and $\mu + k$ and you'll get the same density for every k. The density is symmetric around μ. The expectation must be μ.

Changing the variance



Changing the mean



Scaling Normals

When we scale a normal (multiplying by a constant or adding a constant) we get a normal random variable back!

If $X \sim \mathcal{N}(\mu, \sigma^2)$

Then for $Y = aX + b$, $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Normals are unique in that you get a NORMAL back.

If you multiply a binomial by $3/2$ you don't get a binomial (it's support isn't even integers!)

X $\frac{3}{2}X$

Normals also have the property that if X, Y are independent normals, then $X + Y$ is also a normal.

Normalize

$\sqrt{Var(X)}$ = Standard deviation

To turn $X \sim \mathcal{N}(\mu, \sigma^2)$ into $Y \sim \mathcal{N}(0,1)$ you want to set

$$Y = \frac{X - \mu}{\boxed{\sigma}}$$

Why normalize?

The density is a mess. The CDF does not have a pretty closed form.

But we're going to need the CDF a lot, so…

Table of Standard Normal CDF

2.00

The way we'll evaluate the CDF of a normal is to:

1. convert to a standard normal
2. Round the "z-score" to the hundredths place.
3. Look up the value in the table.

It's 2021, we're using a table?

The table makes sure we have consistent rounding rules (makes it easier for us to debug with you).

You can't evaluate this by hand – the "z-score" can give you intuition right away.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Use the table!

We'll use the notation $\Phi(z)$ to mean $F_X(z)$ where $X \sim \mathcal{N}(0,1)$.

Let $Y \sim \mathcal{N}(5,4)$ what is $\mathbb{P}(Y > 9)$?

$\mathbb{P}(Y > 9)$

$= \mathbb{P}\left(\frac{Y-5}{2} > \frac{9-5}{2}\right)$ we've just written the inequality in a weird way.

$= \mathbb{P}(X > \frac{9-5}{2})$ where X is $\mathcal{N}(0,1)$.

$= 1 - \mathbb{P}\left(X \leq \frac{9-5}{2}\right) = 1 - \Phi(2.00) = 1 - 0.97725 = .02275.$

More practice

Let $X \sim \mathcal{N}(3, 2)$.

What is the probability that $1 \leq X \leq 4$

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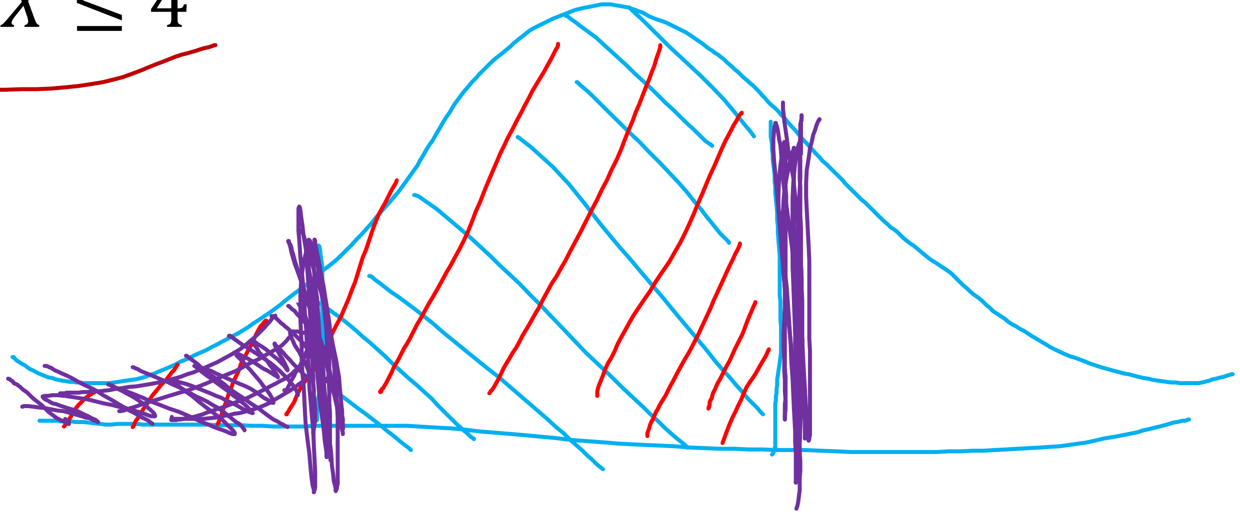
Let $X \sim \mathcal{N}(3, 2)$.

What is the probability that $1 \leq X \leq 4$



$$\mathbb{P}(1 \leq X \leq 4)$$

$$= \mathbb{P}\left(\frac{1-3}{\sqrt{2}} \leq \frac{X-3}{\sqrt{2}} \leq \frac{4-3}{\sqrt{2}}\right)$$

$$\approx \mathbb{P}\left(-1.41 \leq \frac{X-3}{\sqrt{2}} \leq .71\right)$$

$$= \Phi(.71) - \Phi(-1.41)$$

Phi $F_X(\;)$

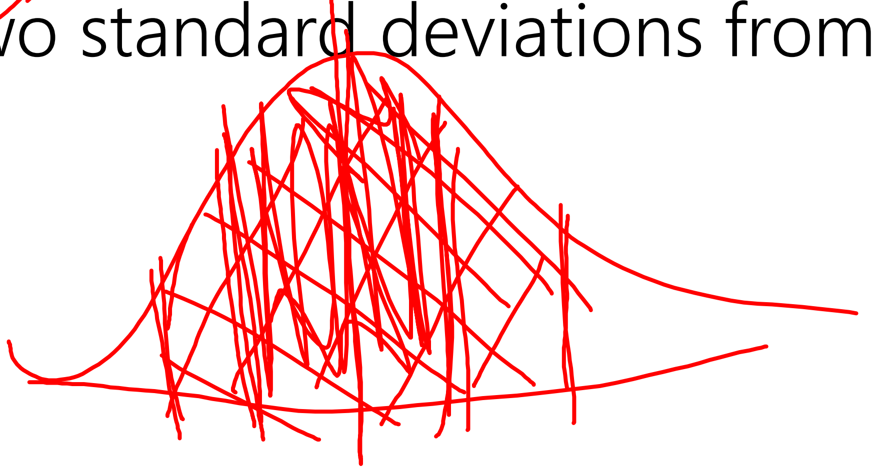
$$= \Phi(.71) - \left(1 - \Phi(1.41)\right) = .76115 - (1 - .92073) = .68188.$$

In real life

$X \sim N(5, 4)$

What's the probability of being at most two standard deviations from the mean?

$P(X \geq 9)$



$$= \Phi(2) - \Phi(-2)$$

$$= \Phi(2) - \left(1 - \Phi(2)\right)$$

$$= .97725 - (1 - .97725) = .9545$$

You'll sometimes hear statisticians refer to the "68-95-99.7 rule" which is the probability of being within 1,2, or 3 standard deviations of the mean.