Section 1: Solutions

Review of Main Concepts (Counting)

- **Sum Rule:** If an experiment can either end up being one of N outcomes, or one of M outcomes (where there is no overlap), then the total number of possible outcomes is $N + M$. e
- **Product Rule:** Suppose events $A_1, ..., A_n$ each have $m_1, ...m_n$ possible outcomes, respectively. Then there are $m_1 \cdot m_2 \cdot m_3 \cdots m_n = \prod_{i=1}^{n} m_i$ possible outcomes overall.
- **Number of ways to order n distinct objects:** $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$
- **Number of ways to select from n distinct objects:**
  - **Permutations** (number of ways to linearly arrange k objects out of n distinct objects, when the order of the k objects matters):

$$P(n, k) = \frac{n!}{(n-k)!}$$

  - **Combinations** (number of ways to choose k objects out of n distinct objects, when the order of the k objects does not matter):

$$\frac{n!}{k!(n-k)!} = \binom{n}{k} = C(n, k)$$

- **Principle of Inclusion-Exclusion (PIE):** 2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
  3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
  In general: +singles - doubles + triples - quads + ...
- **Complementary Counting (Complementing):** If asked to find the number of ways to do X, you can: find the total number of ways and then subtract the number of ways to not do X.
- **Binomial Theorem:** $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}$: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$
- **Union:** The union of two events E and F is denoted $E \cup F$
- **Intersection:** The intersection of two events E and F is denoted $E \cap F$ or EF
- **Mutually Exclusive:** Events E and F are mutually exclusive iff $E \cap F = \emptyset$
- **Complement:** The complement of an event E is denoted E^C or $\overline{E}$ or $\neg E$, and is equal to $\Omega \setminus E$
- **DeMorgan's Laws:** $(E \cup F)^C = E^C \cap F^C$ and $(E \cap F)^C = E^C \cup F^C$

1. Seating

How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if ...

(a) ...all couples are to get adjacent seats?

Solution:

Consider each couple as a unit. Apply the product rule, first choosing one of the $5!$ permutations of the 5 couples, and then, for each couple in turn, choosing one of the 2 permutations for how they sit (for a total of 2^5). Therefore, the answer is: $\boxed{5! \cdot 2^5.}$

(b) …anyone can sit anywhere, except that one couple insists on *not* sitting in adjacent seats?

Solution:

Apply complementary counting to first compute the total number of arrangements of the 10 people, and then subtract from this the number of arrangements in which that particular couple does get adjacent seats. There are $10!$ for the former, and there are $9! \cdot 2$ arrangements in which this couple does sit in adjacent seats, since you can treat the couple as a unit, permute the 9 "individuals" (consisting of 8 people plus the couple) and then consider the 2 permutations for that couple. That means the answer to the question is $\boxed{10! - 9! \cdot 2 = 8 \cdot 9!}$.

Alternatively, we can do casework. Name the two people in the couple A and B. There are two cases: A can sit on one of the ends, or not. If A sits on an end seat, A has 2 choices and B has 8 possible seats. If A doesn't sit on the end, A has 8 choices and B only has 7. So there are a total of $2 \cdot 8 + 8 \cdot 7$ ways A and B can sit. Once they do, the other 8 people can sit in $8!$ ways since there are no other restrictions. Hence the total number of ways is $\boxed{(2 \cdot 8 + 8 \cdot 7)8! = 9 \cdot 8 \cdot 8! = 8 \cdot 9!}$.

2. Weird Card Game

In how many ways can a pack of fifty-two cards be dealt to thirteen players, four to each, so that every player has one card of each suit? **Solution:**

Apply the product rule: First deal the hearts, one to each person, then the spades, one to each person, then diamonds, then the clubs. For each of these steps, there are $13!$ possibilities. Therefore, the answer is $\boxed{13!^4}$.

3. HBCDEFGA

How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

Solution:

The total number of permutations is $8!$. The number of permutations with A at the beginning is $7!$ and the number with H at the end is $7!$. By inclusion/exclusion, the number that have either A at the beginning or H at the end or both is $2 \cdot 7! - 6!$ since there are $6!$ that have A at the beginning and H at the end. Finally, using complementary counting, the number that have neither A at the end or H at the end is $8! - (2 \cdot 7! - 6!)$.

4. Escape the Professor

There are 6 security professors and 7 theory professors taking part in an escape room. If 4 security professors and 4 theory professors are chosen and paired off, how many pairings are possible?

Solution:

$\boxed{\binom{6}{4}\binom{7}{4}4!}$. Apply the product rule to first choose 4 of the security professors, then 4 of the theory professors. Then assign each theory professor to a security professor (4 choices for the first, 3 for the second and so on).

5. Birthday Cake

A chef is preparing desserts for the week, starting on a Sunday. On each day, only one of five desserts (apple pie, cherry pie, strawberry pie, pineapple pie, and cake) may be served. On Thursday there is a birthday, so cake must be served that day. On no two consecutive days can the chef serve the same dessert. How many dessert menus are there for the week?

Solution:

Apply the product rule. Start from Thursday and work forward and backward in the week:

More precisely, given the 1 choice on Thursday, for each of Wednesday and Friday, there are 4 choices (the different pie options). Given the choice on Wednesday, there are 4 choices for Tuesday, and given the choice on Tuesday, there are 4 choices for Monday, and given the choice on Monday, there are 4 choices on Sunday. Similarly, given the choice on Friday, there are 4 choices on Saturday.

Therefore the answer is $4 \cdot 4 \cdot 4 \cdot 4 \cdot 1 \cdot 4 \cdot 4 = \boxed{4^6}$

6. Full Class

There are 40 seats and 40 students in a classroom. Suppose that the front row contains 10 seats, and there are 5 students who must sit in the front row in order to see the board clearly. How many seating arrangements are possible with this restriction?

Solution:

Use the product rule: First choose the seats that the five students who must sit in the front row will take and then choose their placement in these seats. For this, there are $\binom{10}{5} \cdot 5!$ choices. Then arrange the remaining 35 students in the remaining 35 seats (35! ways).

So the answer is $\boxed{\binom{10}{5} \cdot 5! \cdot 35!}$.

7. Paired Finals

Suppose you are to take a CSE 312 final in pairs. There are 100 students in the class and 8 TAs, so 8 lucky students will get to pair up with a TA. Each TA must take the exam with some student, but two TAs cannot take the exam together. How many ways can they pair up? **Solution:**

Apply the product rule. First we choose the 8 lucky students, and then pair them with a TA. There are $\binom{100}{8}$ ways to choose the students, and $8!$ ways to pair them with the TAs. There are 92 students left. The first one has 91 choices. Then there are 90 students left. The next one has 89 choices. And so on.

So the total number of ways is $\boxed{\binom{100}{8} \cdot 8! \cdot 91 \cdot 89 \cdot \ldots \cdot 3 \cdot 1}$.

8. Photographs

Suppose that 8 people, including you and a friend, line up for a picture. In how many ways can the photographer organize the line if she wants to have fewer than 2 people between you and your friend?

Solution:

This is most easily solved using the sum rule. Count the number of ways the line can be organized if you are next to your friend. Then count the number of ways the line can be organized if there is one person between you and your friend. Then use the sum rule to add these up.

Case 1: You are next to your friend. So we can think of you and your friend as being a ”unit”. Now apply the product rule: there are $7!$ ways to arrange the other 6 people together with the unit (of you and your friend). Once arranged, there are 2 ways to rearrange you and your friend in the order. So there are $7! \cdot 2$ ways to to line people up if you are next to your friend.

Case 2: There is exactly 1 person between you and your friend. Apply the product rule by first picking the person who is between you (6 choices). Then, thinking of you, your friend and that person as a ”unit”, consider all arrangements of the 5 people plus the unit ($6!$ ways). Finally, there are two ways for you and your friend to be placed within the trio. Therefore, altogether there are $6 \cdot 6! \cdot 2$ possibilities.

Therefore, the final answer is $\boxed{(2 \cdot 7 + 2 \cdot 6) \cdot 6!}$

9. Rabbits!

Rabbits Peter and Pauline have three offspring: Flopsie, Mopsie, and Cotton-tail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

Solution:

Find the number of possibilities when Peter and Pauline go to the same store, and find the number of possibilities when they go to different stores, and then use the sum rule to get the final answer.

If Peter and Pauline go to the same store, there are 4 stores it could be. For each such choice, there are 3 choices of store for each of the 3 offspring, so 3^3 choices for all the offspring.

If Peter and Pauline go to different stores, there are $4 \cdot 3 = 12$ pairs of stores they could go to. For each such choice, there are 2 choices of store for each of the 3 offspring, so 2^3 choices for all the offspring.

Therefore the answer is $\boxed{4 \cdot 3^3 + 12 \cdot 2^3}$.

10. Extended Family Portrait

A group of n families, each with m members, are to be lined up for a photograph. In how many ways can the nm people be arranged if members of a family must stay together? **Solution:**

Apply the product rule. First order the families; there are $n!$ ways to do this. Then consider the families one by one and reorder their members. Within each family, there are $m!$ ways to order their members. So there are a total of $\boxed{n!(m!)^n}$ ways to line these people up according to the given constraints.

11. Subsubset

Let $[n] = \{1, 2, ..., n\}$ denote the first n natural numbers. How many (ordered) pairs of subsets (A, B) are there such that $A \subseteq B \subseteq [n]$?

Solution:

There are two ways to do this question:

First way: Apply the sum rule by adding up the number of ways of doing this where B has size k, where k is any integer between 0 and n. Now apply the product rule to find the number of ways to choose B of size exactly k (there are $\binom{n}{k}$ possibilities for B), and then once B is selected, count the number of ways of choosing A which has to be a subset of B (2^k ways). Hence the number of such ordered pairs of subsets is

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} = 3^n$$

by the Binomial Theorem.

Second way: Realize that, if there are no restrictions, for each element i of $1, ..., n$, there are four possibilities: it can be in only A, only B, both, or neither. In our case, there is only one that is not valid (violates $A \subseteq B$): being in A but not B. Hence there are 3 choices for each element, so the total number of such ordered pairs of subsets is 3^n.

12. Divide Me

How many numbers in $[360]$ are divisible by:

(a) 4, 6, and 9? **Solution:**

This is just the multiples of $\text{lcm}(4, 6, 9) = 36$. There are $\boxed{\frac{360}{36} = 10}$ multiples.

(b) 4, 6, or 9? **Solution:**

Use inclusion-exclusion. $\boxed{\frac{360}{4} + \frac{360}{6} + \frac{360}{9} - \frac{360}{\text{lcm}(4,6)} - \frac{360}{\text{lcm}(4,9)} - \frac{360}{\text{lcm}(6,9)} + \frac{360}{\text{lcm}(4,6,9)}}$

(c) Neither 4, 6, nor 9? **Solution:**

This is just the complement of the previous part, so it is 360 minus the answer to (b).

13. A Team and a Captain

Give a combinatorial proof of the following identity:

$$n\binom{n-1}{r-1} = \binom{n}{r} r.$$

Hint: Consider two ways to choose a team of size r out of a set of size n and a captain of the team (who is also one of the team members). **Solution:**

Remember that a combinatorial proof just requires that we show both sides are equivalent ways of counting a situation.

Left hand side: Choose a team of size r and a captain for that team (from among the r) by first choosing the captain (n choices) and then choosing the rest of the team $\binom{n-1}{r-1}$.

Right hand side: Choose a team of size r and a captain for that team by first choosing the team ($\binom{n}{r}$ choices) and then choosing the captain from among the members of the team (r choices).

14. Subsubset

Let $[n] = \{1, 2, ..., n\}$ denote the first n natural numbers. How many (ordered) pairs of subsets (A, B) are there such that $A \subseteq B \subseteq [n]$?

Solution:

There are two ways to do this question:

First way: Apply the sum rule by adding up the number of ways of doing this where B has size k, where k is any integer between 0 and n. Now apply the product rule to find the number of ways to choose B of size exactly k (there are $\binom{n}{k}$ possibilities for B), and then once B is selected, count the number of ways of choosing A which has to be a subset of B (2^k ways). Hence the number of such ordered pairs of subsets is

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} = 3^n$$

by the Binomial Theorem.

Second way: Realize that, if there are no restrictions, for each element i of $1, ..., n$, there are four possibilities: it can be in only A, only B, both, or neither. In our case, there is only one that is not valid (violates $A \subseteq B$): being in A but not B. Hence there are 3 choices for each element, so the total number of such ordered pairs of subsets is 3^n.

15. GREED INNIT

Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other. For example, "INGREEDINT" is invalid because the two E's are adjacent.

Solution:

We use inclusion-exclusion. Let Ω be the set of all anagrams (permutations) of "INGREDIENT", and A_I be the set of all anagrams with two consecutive I's. Define A_E and A_N similarly. $A_I \cup A_E \cup A_N$ clearly are the set of anagrams we don't want. So we use complementing to count the size of $\Omega \setminus (A_I \cup A_E \cup A_N)$. By inclusion exclusion, $|A_I \cup A_E \cup A_N|$ =singles-doubles+triples, and by complementing, $|\Omega \setminus (A_I \cup A_E \cup A_N)| = |\Omega| - |A_I \cup A_E \cup A_N|$.

First, $|\Omega| = \dfrac{10!}{2!2!2!}$ because there are 2 of each of I,E,N's (multinomial coefficient). Clearly, the size of A_I is the same as A_E and A_N. So $|A_I| = \dfrac{9!}{2!2!}$ because we treat the two adjacent I's as one entity. We also need $|A_I \cap A_E| = \dfrac{8!}{2!}$ because we treat the two adjacent I's as one entity and the two adjacent E's as one entity (same for all doubles). Finally, $|A_I \cap A_E \cap A_N| = 7!$ since we treat each pair of adjacent I's, E's, and N's as one entity.

Putting this together gives $\boxed{\frac{10!}{2!2!2!} - \left(\binom{3}{1} \cdot \frac{9!}{2!2!} - \binom{3}{2} \cdot \frac{8!}{2!} + \binom{3}{3} \cdot 7!\right)}$

Repeat the question for the letters “AAAAABBB”. **Solution:**

Note that no A’s and no B’s can be adjacent. So let us put the B’s down first: $\_B\_B\_B\_$

By the pigeonhole principle, two A’s must go in the same slot, but then they would be adjacent, so there are $\boxed{\text{no ways}}$.