Please download the activity slide for today!

Its named "pdf".

More Counting

CSE 312 Summer 21
Lecture 2

Announcements

Office hours for this week:

Kushal: Friday 1 pm – 2pm

Student Information for Office Hours

https://forms.gle/S5eDFe5bgipgh2MC8

Just one question – your current time zone

Syllabus should be up by tonight

Where Are We?

Last Lecture:

Sum and Product Rules

Sequential Process

Representation is important!

Today:

Permutations and Combinations

Complementary Counting

Binomial Theorem

More sequential process practice

How many length 3 sequences are there consisting of distinct elements of $\{1,2,3\}$?

Pause

Questions in combinatorics are probability are often dense.

A single word can totally change the question.

- Does order matter? $12 \leftrightarrow 21$ $\{1,2\}$
- Are repeats allowed? AD_1D_2
- What makes two things "count the same" or "count as different"?

AD_2D_1

More sequential process practice

Let's look for some keywords in the question

How many length 3 sequences are there consisting of distinct elements of {1,2,3}?

1, 2, 3 ⟷ 3, 1, 2

123 ≠ 312

- Sequence implies that order matters – (1,2,3) and (3,1,2) are different
- Distinct implies that we cannot repeat elements – (1,2,1) does not count {3, 1, 1}
- {1,2,3} is our "universe" – our set of allowed elements

{1, 2, 4}

$$\left(\underline{3}, \underline{2}, \underline{1} \right) = 3 \cdot 2 \cdot 1 = 6$$

More sequential process practice

How many length 3 sequences are there consisting of distinct elements of $\{1,2,3\}$?

Step 1: 3 options for the first element

Step 2: 2 (remaining) options for the second element

Step 3: 1 (remaining) option for the third element

$n! = n \cdot (n-1) \cdot (n-2) \cdots 1$

$3 \cdot 2 \cdot 1 = 6$

Factorial

This formula shows up a lot!

The number of ways to "permute" (i.e., "reorder" or "list without repeats") n elements is "n factorial"

$\boldsymbol{n}$ factorial

$$\boldsymbol{n! = n \cdot (n-1) \cdot (n-2) \cdots 1}$$

We only define $n!$ for natural numbers n.

As a convention, we define: $0! = 1$.

Distinct Letters

26^5 $\underline{26}\ \underline{26}\ \underline{26}\ \underline{26}\ \underline{26}$

How many strings of length 5 are there over the alphabet $\{A, B, C, \ldots, Z\}$ where each string does not repeat a letter?

E.g., SWING is allowed but SALSA is not

$\underline{26}\ \underline{25}\ \underline{24}\ \underline{23}\ \underline{22}$

S W I N G

Distinct Letters

How many strings of length 5 are there over the alphabet $\{A, B, C, \ldots, Z\}$ where each string does not repeat a letter?

E.g., **SWING** is allowed but **SALSA** is not

Step 1: 26 options for the first letter

Step 2: 25 options for the second letter

…

Step 5: 22 options for the fifth letter

$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22$

In General

k-permutation

The number of k-element sequences of distinct symbols from a universe of n symbols is:

$$P(n,k) = n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}$$

Said out loud as "P n k" or "n permute k" or "n pick k"

$$\frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

Alternative notation: P_k^n or ${}_nP_k$

Edge cases: $P(n,n) = n!, P(n,0) = 1, P(n,k)$ is undefined for $k < 0$ or $k > n$

Change it slightly

A, A $\Rightarrow$ {A}

{A, B} = {B, A}

How many subsets of size 5 are there of $\{A, B, \ldots, Z\}$?

Remember for sets we don't count repeats – so that constraint still stays

But for sets order does not matter

$\{S, W, I, N, G\}$ is the same as $\{W, I, N, G, S\}$ even though "SWING" and "WINGS" are different strings.

Count two ways

$$\frac{26!}{5!\,21!}$$

Let's artificially introduce a requirement that we are supposed to have an ordered list.

Then the total is $P(26, 5)$ as we have seen before.

$$\frac{26!}{21!}$$

How else could we get an ordered list? With a sequential process:

Step 1: Choose a subset ?

Step 2: Put the subset in order $n! = 5!$

These better give the same number, so:

$$\frac{26!}{(26-5)!} = ? \cdot 5!$$

$$\Rightarrow \quad ? = \frac{26!}{5!\,(26-5)!} = \frac{26!}{5!\,21!}$$

Count two ways

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How else could we get an ordered list? With a sequential process:

Step 1: Choose a subset

Step 2: Put the subset in order

These better give the same number, so:

$$\frac{26!}{(26-5)!} = ? \cdot 5!$$

So, the number of size 5 subsets of the size 26 set is:

$$\frac{26!}{(26-5)!5!}$$

Number of subsets

k-combination

The number of k-element subsets from a set of n symbols is:

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!\,(n-k)!}$$

Said out loud "n choose k" (or sometimes: "n combination k")

Notations:

${}_nC_k$ or $\binom{n}{k}$ or $C(n,k)$ all mean "number of size k subsets of a size n set

Edge cases: $\binom{n}{0} = 1$, $\binom{n}{n} = 1$, $\binom{n}{k}$ is undefined for $k < 0$ or $k > n$

Takeaway

$$\underbrace{a \quad b \quad d \quad f \quad g}_{5!} \quad | \quad \underbrace{i \quad j \quad k \quad \ldots \quad z}_{21!}$$

The second way of counting hints at a generally useful trick:

Pretend that order does matter, then divide by the number of orderings of the parts where order does not matter.

$$P(n,k) = \frac{26!}{21!}$$

For example, here is another way to get the formula for combinations:

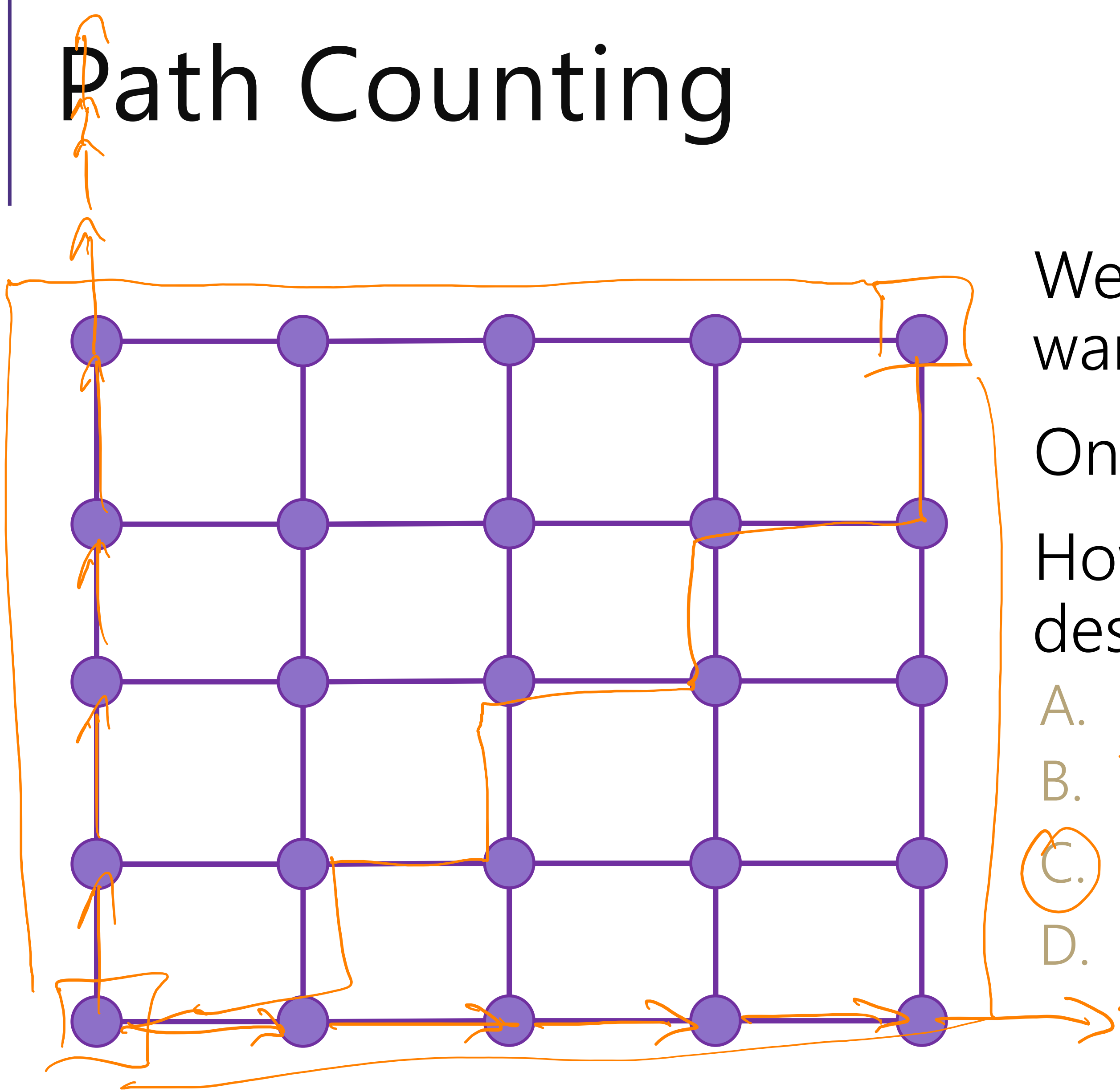
You have n elements. Put them in order, take k as your set.

$n!$ orderings overall. We have overcounted because:

- Among the first k, order does not matter. Divide by $k!$
- Among the last $(n-k)$, order does not matter. Divide by $(n-k)!$

$$\frac{n!}{(n-k)!\,k!}$$

Path Counting



We are in the lower left corner and want to get to the upper right corner.

Only moves allowed are up and right.

How many paths lead to our destination?

A. $\boxed{2^8}$ ✗

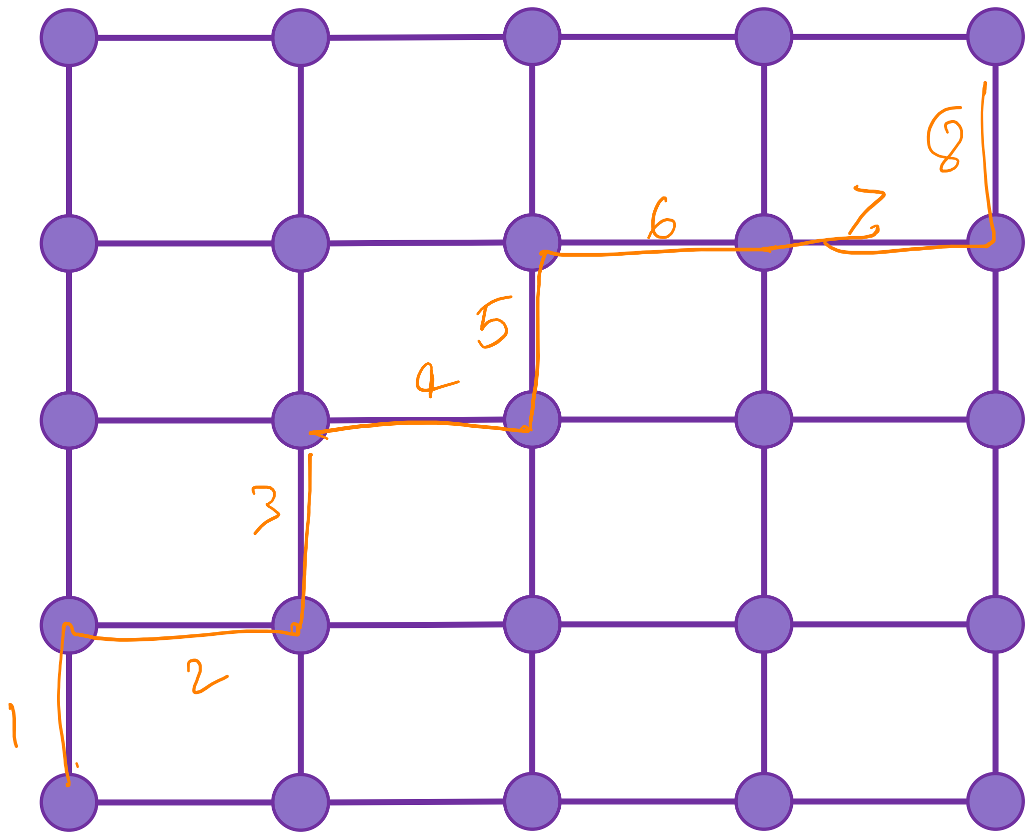
B. $P(8,4)$ ✗ (1, 2, 7, 8)

C. $\binom{8}{4}$

D. Something else

Fill out the Poll Everywhere for Kushal to adjust his explanation
Go to pollev.com/cse312su21

Path Counting



We are in the lower left corner and want to get to the upper right corner.

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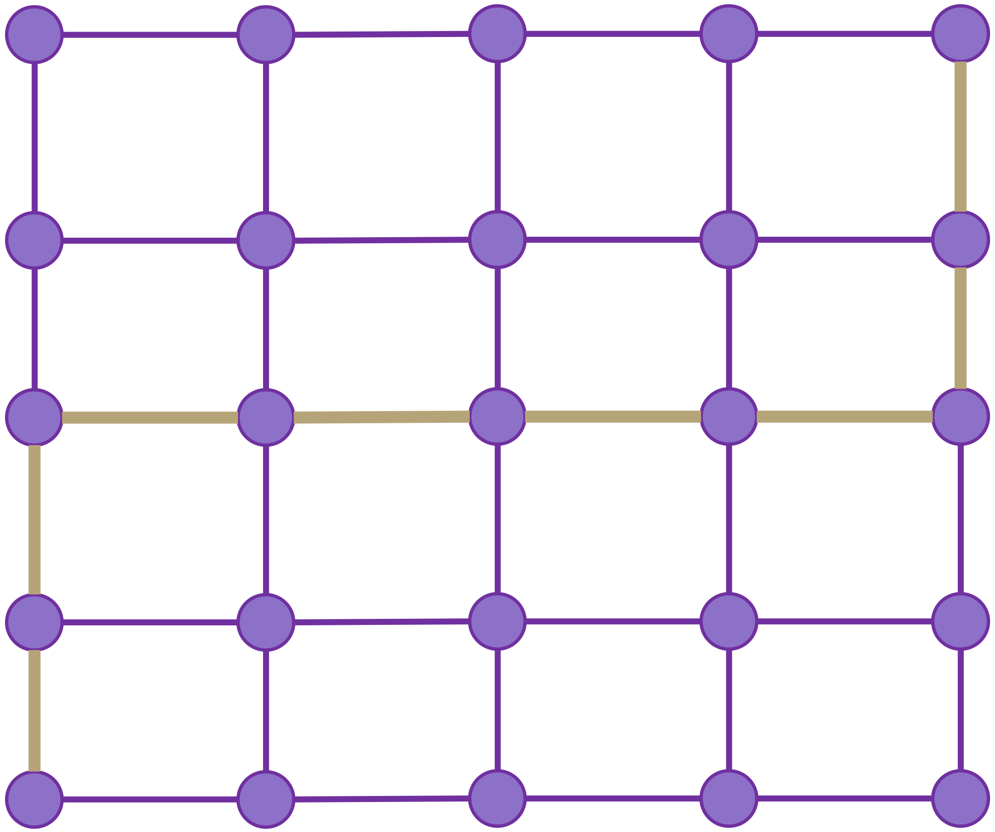
How many paths lead to our destination?

Idea 1:

We are going to take 8 steps

Choose which SET of 4 of the steps will be up (the others will be right).

Path Counting



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How many paths lead to our destination?

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Choose which SET of 4 of the steps will be up (the others will be right).

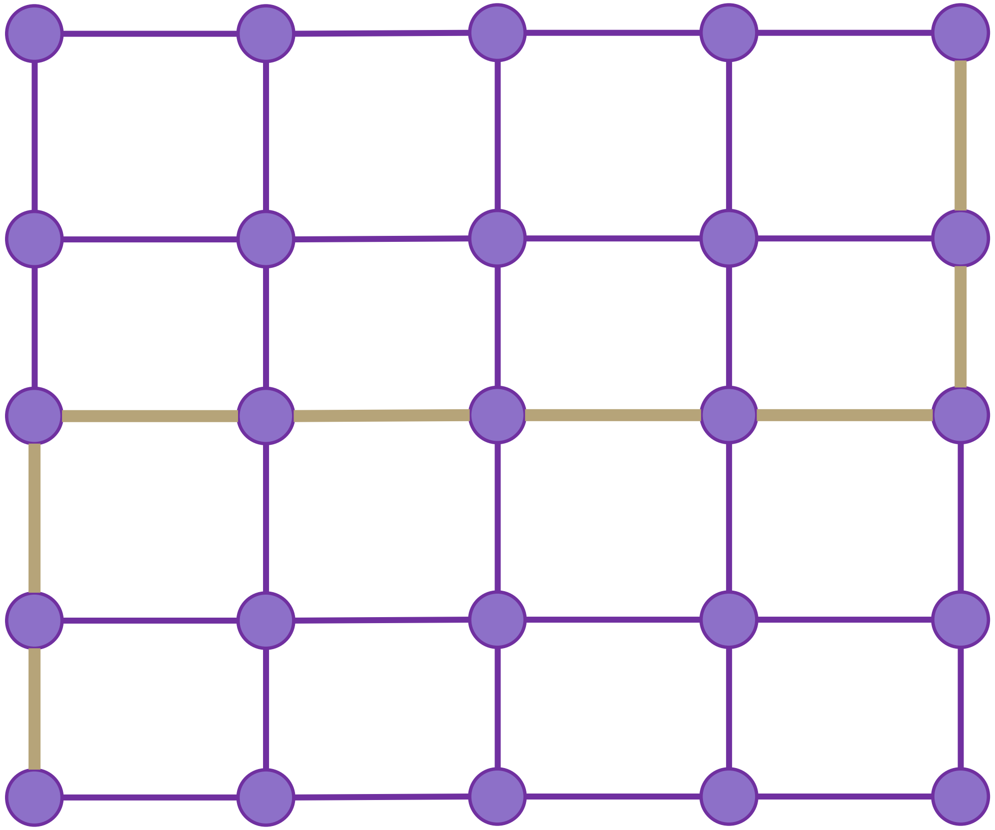
E.g. {1,2,7,8}

=

{8, 7, 2, 1}

$\binom{8}{4}$

Path Counting



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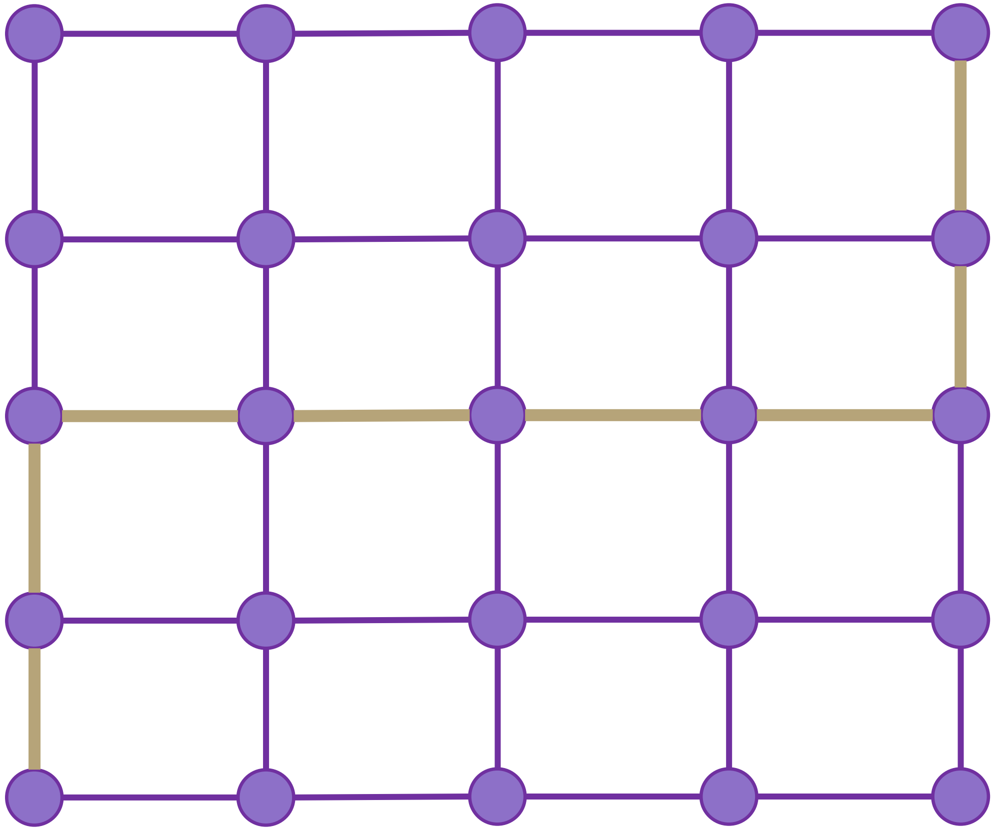
E.g. $\{1,2,7,8\}$

Hence, how many size 4 subsets are there?

Path Counting



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We are going to take 8 steps

Choose which SET of 4 of the steps will be up (the others will be right).

E.g. $\{1,2,7,8\}$

Hence, how many size 4 subsets are there?

The answer: $\binom{8}{4}$

Path Counting

How do we know we are done counting?

Why did we not count the steps to the right?

$$\binom{8}{4} \cdot \binom{4}{4}$$

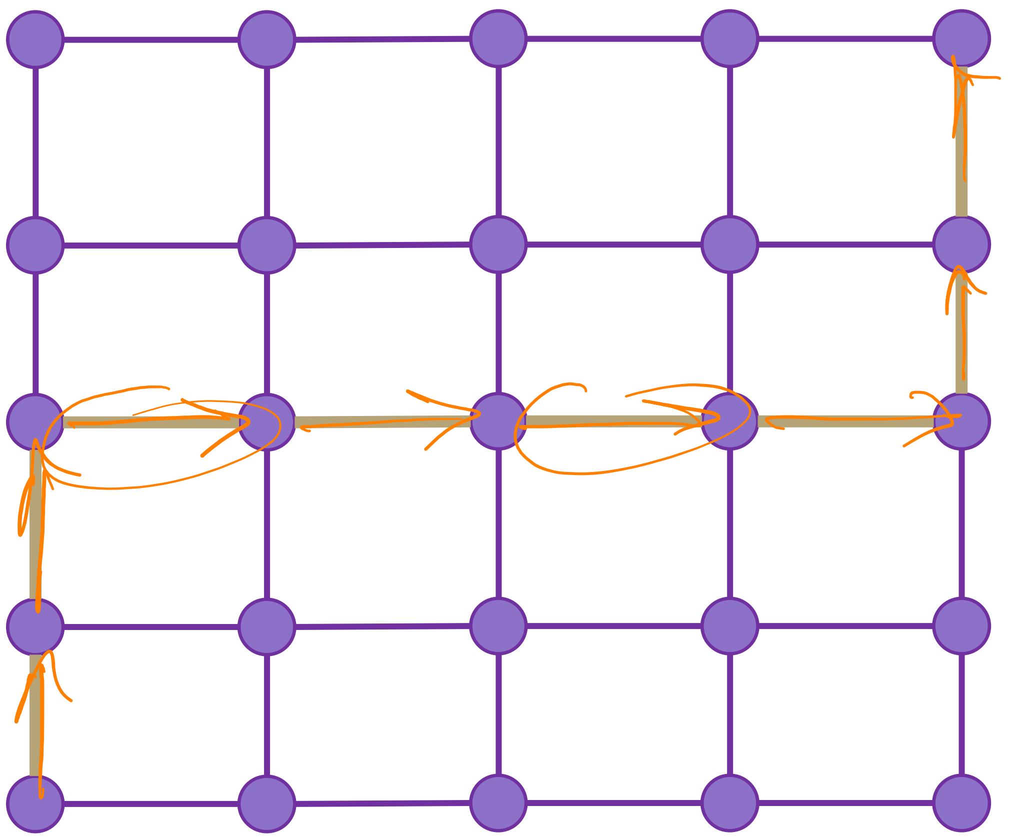
We have chosen the 4 steps up. Of the remaining 4 steps, we will choose 4 of them to be to the right.

The number of ways we choose steps to the right: $\binom{4}{4} = 1\ way$

So, multiplying by this does not change anything.

We are done counting when given the choices of our sequential process, we know exactly which path we take. And given a path, we know exactly the choices of our sequential process.

Path Counting



We are in the lower left corner and want to get to the upper right corner.

Only moves allowed are up and right.

How many paths lead to our destination?

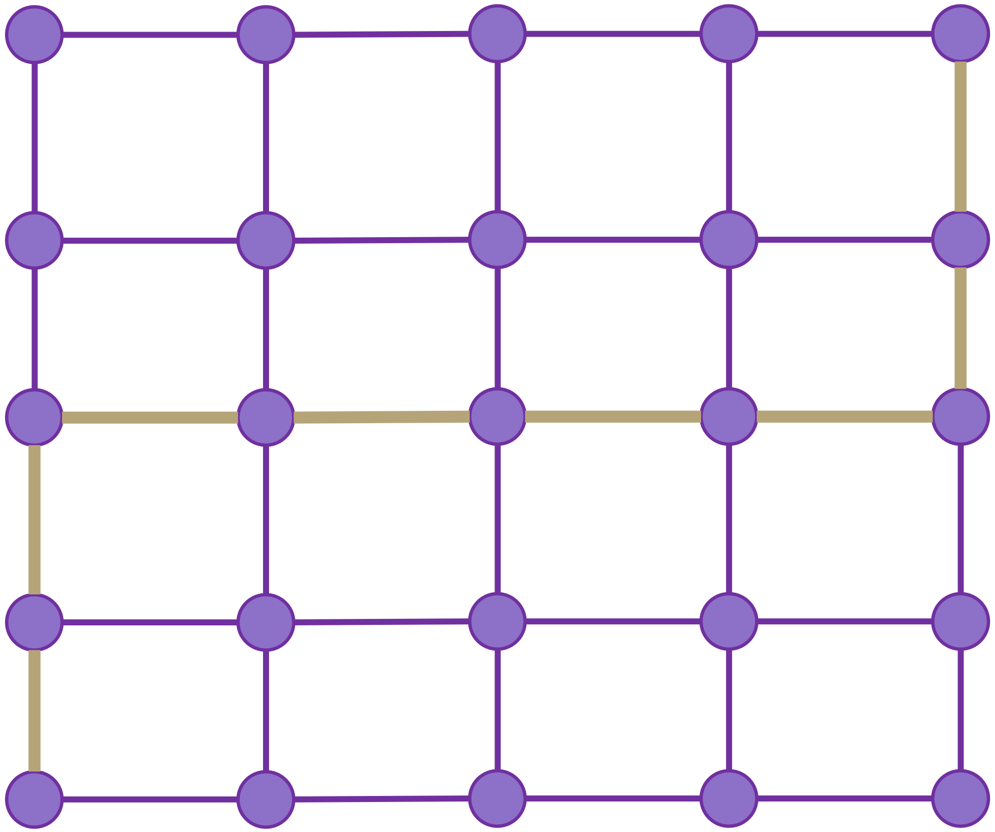
Idea 2: Introduce artificial ordering

Order $\underline{\uparrow_A}\,\underline{\uparrow_B}\rightarrow_A\rightarrow_B\rightarrow_C\rightarrow_D\uparrow_C\uparrow_D$ 8!

$$\frac{8!}{4!\,4!}$$

$\rightarrow_A \rightarrow_C$

Path Counting



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How many paths lead to our destination?

Idea 2: Introduce artificial ordering

Order $\uparrow_A \uparrow_B \rightarrow_A \rightarrow_B \rightarrow_C \rightarrow_D \uparrow_C \uparrow_D$ $8!$

Remove the overcounting

The 4 $\uparrow$ are the really the same, divide by $4!$

The 4 $\rightarrow$ are the really the same, divide by $4!$

Total: $\dfrac{8!}{4! \cdot 4!} = \binom{8}{4}$

Overcounting

How many anagrams are there of SEATTLE?

SEATTLE

SAETTLE

Anagram – a rearrangement of letters

7!

Overcounting

How many anagrams are there of SEATTLE?

Anagram – a rearrangement of letters

It is not 7!

7! counts SEATTLE and SEATTLE (where the Es are switched) as different words.

$$\frac{7!}{2!\,2!}$$

GODADDY

$$\frac{7!}{3!}$$

Overcounting

How many anagrams are there of SEATTLE?

Pretend that the order of the Es and Ts relative to each other matter (SEATTLE and SEATTLE are indeed different)

How many arrangements of SEATTLE? 7!

Now, returning to the question where both Es (and Ts) are the same

How have we overcounted?

Es relative to each other and Ts relative to each other.

Overcounts: $2! \cdot 2!$

$\binom{7}{2,2}$

Final Answer: $\frac{7!}{2! \cdot 2!}$

One More Counting Technique

Complementary Counting

Count the complement of the set you're interested in.

$\Rightarrow$ 1a , 2a , 3a , 4a , 5a

How many length 5 strings over $\{a, b, c, \ldots, z\}$ are there with **at least** $1\ a$?

Let A be the set of strings we're interested in, $\mathcal{U}$ be all length 5 strings

$\bar{A}$ will therefore be the set of strings that have no a in them.

$|A| = \left|\mathcal{U} \setminus \bar{A}\right| = |\mathcal{U}| - \left|\bar{A}\right| = 26^5 - 25^5$

Takeaways

Formulas for factorial, permutations, and combinations.

( ) · arrangements

A useful trick for counting is to pretend that order matters, then account for the overcounting at the end – dividing the repetitions.

When you are trying to prove facts about counting (more in the appendix), try to have each side of the equation count the same thing.
It will be much more intuitive and fun than churning through complex algebra.

Binomial Theorem

In high school you probably memorized

$(x+y)^2 = x^2 + 2xy + y^2$

And $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$\binom{3}{3} \quad \binom{3}{2} \quad \binom{3}{1} \quad \binom{3}{0}$

The Binomial Theorem gives us the equation for every power n:

The Binomial Theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Some intuition

$(x+y)^n = (x+y)(x+y) \cdots \cdots (x+y)$

$\# x + \# y = n$

The Binomial Theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Intuition: Every monomial on the right-hand-side has either x or y from each of the terms on the left.

How many copies of $x^i y^{n-i}$ do you get?

Well how many ways are there to choose i x's and $n-i$ y's? $\binom{n}{i}$. $\binom{n-i}{n-i}$

Formal proof? Induction!

So What?

Well…if you saw it before, now you have a better understanding now of why it's true.

There are also a few cute applications of the binomial theorem to proving other theorems (usually by plugging in numbers for x and y)

$x + y$

For example, set $x = 1$ and $y = 1$ then

$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i}$.

i.e. if you sum up binomial coefficients, you get 2^n.

Exercise: reprove this equation (directly) with a combinatorial proof (where have we seen 2^n recently?)

Appendix: Combination Facts

Some Facts about combinations

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

Symmetry of combinations: $\binom{n}{k} = \binom{n}{n-k}$

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$\binom{n}{n-k} = \frac{n!}{(n-(n-k))!\,(n-k)!}$$

$$= \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

Two Proofs of Symmetry

Proof 1: By algebra

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$ Definition of Combination

$$= \frac{n!}{(n-k)!k!}$$ Algebra (commutativity of multiplication)

$$= \binom{n}{n-k}$$ Definition of Combination

Two Proofs of Symmetry

Wasn’t that a great proof.

Airtight. No disputing it.

Got to say “commutativity of multiplication.”

But…do you know *why*? Can you *feel* why it’s true?

Two Proofs of Symmetry

Suppose you have n people, and need to choose k people to be on your team. We will count the number of possible teams two different ways.

Way 1: We choose the k people to be on the team. Since order doesn't matter (you're on the team or not), there are $\binom{n}{k}$ possible teams.

Way 2: We choose the $n - k$ people to NOT be on the team. Everyone else is on it. Since order again doesn't matter, there are $\binom{n}{n-k}$ possible ways to choose the team.

Since we're counting the same thing, the numbers must be equal.
So $\binom{n}{k} = \binom{n}{n-k}$.

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$
\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-1-[k-1])!} + \frac{(n-1)!}{k!(n-1-k)!} && \text{definition of combination} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} && \text{subtraction} \\
&= \frac{[(n-1)!k!(n-k-1)!] + [(n-1)!(k-1)!(n-k)!]}{k!(k-1)!(n-k)!(n-k-1)!} && \text{Find a common denominator} \\
&= \frac{(n-1)!(k-1)!(n-k-1)!\,[k+(n-k)]}{k!(k-1)!(n-k)!(n-k-1)!} && \text{factor out common terms} \\
&= \frac{(n-1)!\,[k+(n-k)]}{k!(n-k)!} && \text{Cancel } (k-1)!(n-k-1)! \\
&= \frac{(n-1)!\cdot n}{k!(n-k)!} = \frac{n!}{k!(n-k)!} && \text{Algebra} \\
&= \binom{n}{k} && \text{Definition of combination}
\end{aligned}
$$

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

You and $n-1$ other people are trying out for a k person team. How many possible teams are there?

1) You are in!! $(k-1)$ other people

$$\binom{n-1}{k-1}$$

$$+$$

2) You are not ☹

$$\binom{n-1}{k}$$

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

You and $n-1$ other people are trying out for a k person team. How many possible teams are there?

Way 1: There are n people total, of which we're choosing k (and since it's a team order doesn't matter) $\binom{n}{k}$.

Way 2: There are two types of teams. Those for which you make the team, and those for which you don't.

If you do make the team, then $k-1$ of the other $n-1$ also make it.

If you don't make the team, k of the other $n-1$ also make it.

Overall, by sum rule, $\binom{n-1}{k-1} + \binom{n-1}{k}$.

Since we're computing the same number two different ways, they must be equal. So: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

More Practice

Books, revisited

Remember the books problem from lecture 1? Books 1,2,3,4,5 need to be assigned to Alice, Claris, and Pascal (each book to exactly one person).

Now that we know combinations, try a sequential process approach. It won't be as nice as the change of perspective, but we can make it work.

Break into cases based on how many books Alice gets, use the sum rule to combine.

Books, revisited

Step 1: give Alice gets 0 books (1 way to do this)

Step 2: give Claris a subset of the remaining books 2^5 ways.

Step 3: give Pascal the remaining books (no choice – 1 way)

$+$

Step 1: give Alice 1 book ($\binom{5}{1}$ ways to do this)

Step 2: give Claris a subset of the 4 remaining books 2^4 ways.

Step 3: give Pascal the remaining books (no choice – 1 way)

$+ \ \ldots$

Books, revisited

Add all the options together

$$1 \cdot 2^5 \cdot 1 + \binom{5}{1} \cdot 2^4 \cdot 1 + \binom{5}{2} \cdot 2^3 \cdot 1 + \binom{5}{3} \cdot 2^2 \cdot 1 + \binom{5}{4} \cdot 2^1 \cdot 1 + \binom{5}{5} \cdot 2^0 \cdot 1$$

If you plug and chug, you'll get the number we got last time. It took quite a bit of work, but we got there!