Quiz Section 4

Review

1) Probability Mass. For every random variable X, we have $\sum_x \mathbb{P}(X = x) =$ ________.

2) Expectation. $\mathbb{E}[X] =$ ________.

3) Linearity of expectation. For any random variables $X_1, \ldots, X_n$, and real numbers $a_1, \ldots, a_n$,

$$\mathbb{E}[a_1X_1 + \cdots + a_nX_n] = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ .$$

4) Variance. $\text{Var}(X) = \mathbb{E}[(X - \mathbb{E})^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ $\quad \text{Var}(aX + b) = \_\_\_\text{Var}(X)$.

5) Independence. Two random variables X and Y are **independent** if ____________________.

6) Variance and Independence. For any two independent random variables X and Y, $\text{Var}(X + Y) =$ ____________________

Task 1 – Identify that range!

Identify the support/range Ω_X of the random variable X, if X is...

a) The sum of two rolls of a six-sided die.

b) The number of lottery tickets I buy until I win it.

c) The number of heads in n flips of a coin with $0 < \mathbb{P}(\text{head}) < 1$.

d) The number of heads in n flips of a coin with $\mathbb{P}(\text{head}) = 1$.

Task 2 – Symmetric Difference

Suppose A and B are random, independent (possibly empty) subsets of $\{1, 2, \ldots, n\}$, where each subset is equally likely to be chosen as A or B. Consider $A\Delta B = (A \cap B^C) \cup (B \cap A^C) = (A \cup B) \cap (A^C \cup B^C)$, i.e., the set containing elements that are in exactly one of A and B. Let X be the random variable that is the size of $A\Delta B$. What is $\mathbb{E}[X]$?

Task 3 – Hungry Washing Machine

You have 10 pairs of socks (so 20 socks in total), with each pair being a different color. You put them in the washing machine, but the washing machine eats 4 of the socks chosen at random. Every subset of 4 socks is equally probable to be the subset that gets eaten. Let X be the number of complete pairs of socks that you have left.

a) What is the range of X, Ω_X (the set of possible values it can take on)? What is the probability mass function of X?

b) Find $\mathbb{E}[X]$ from the definition of expectation.

c) Find $\mathbb{E}[X]$ using linearity of expectation.

d) Which way was easier? Doing both (a) and (b), or just (c)?

Task 4 – Balls in Bins

Let X be the number of bins that remain empty when m balls are distributed into n bins randomly and independently. For each ball, each bin has an equal probability of being chosen. (Notice that two bins being empty are not independent events: if one bin is empty, that decreases the probability that the second bin will also be empty. This is particularly obvious when $n = 2$ and $m > 0$.) Find $\mathbb{E}[X]$.

Task 5 – Frogger

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_1, to the left with probability p_2, and doesn't move with probability p_3, where $p_1 + p_2 + p_3 = 1$. After 2 seconds, let X be the location of the frog.

a) Find $p_X(k)$, the probability mass function for X.

b) Compute $\mathbb{E}[X]$ from the definition.

c) Compute $\mathbb{E}[X]$ again, but using linearity of expectation.

Task 6 – 3-sided Die

Let the random variable X be the sum of two independent rolls of a fair 3-sided die. (If you are having trouble imagining what that looks like, you can use a 6-sided die and change the numbers on 3 of its faces.)

a) What is the probability mass function of X?

b) Find $\mathbb{E}[X]$ directly from the definition of expectation.

c) Find $\mathbb{E}[X]$ again, but this time using linearity of expectation.

d) What is $\mathrm{Var}(X)$?

Task 7 – Practice

a) Let X be a random variable with $p_X(k) = ck$ for $k \in \{1, \ldots, 5\} = \Omega_X$, and 0 otherwise. Find the value of c that makes X follow a valid probability distribution and compute its mean and variance ($\mathbb{E}[X]$ and $\mathrm{Var}(X)$).

b) Let X be *any* random variable with mean $\mathbb{E}[X] = \mu$ and variance $\mathrm{Var}(X) = \sigma^2$. Find the mean and variance of $Z = \dfrac{X - \mu}{\sigma}$. (When you're done, you'll see why we call this a "standardized" version of X!)

c) Let X, Y be independent random variables. Find the mean and variance of $X - 3Y - 5$ in terms of $\mathbb{E}[X], \mathbb{E}[Y], \mathrm{Var}(X)$, and $\mathrm{Var}(Y)$.

d) Let $X_1, \ldots, X_n$ be independent and identically distributed (iid) random variables each with mean μ and variance σ^2. The sample mean is $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$. Find the mean and variance of $\bar{X}$. If you use the independence assumption anywhere, **explicitly label** at which step(s) it is necessary for your equalities to be true.

Task 8 – Expectations, Independence, and Variance

a) Let U be a random variable which is uniform over the set $[n] = \{1, 2, \ldots, n\}$, i.e, $\mathbb{P}(U = i) = \frac{1}{n}$ for all $i \in [n]$. Compute $\mathbb{E}\left[U^2\right]$ and $\mathrm{Var}(U)$.

Hint: $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.

b) Let Y_1 and Y_2 be the independent outcomes of two dice rolls, and let $Z = Y_1 + Y_2$. Then, compute $\mathbb{E}\left[Z^2\right]$ and $\mathrm{Var}(Z)$.

Hint: Try to use an indirect solution using linearity and independence, without the need of explicitly giving the distribution of Z^2.