Quiz Section 7 – Solutions

Review

1) Central Limit Theorem (CLT): Let $X_1, \ldots, X_n$ be iid random variables with $\mathbb{E}[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$. Let $X = \sum_{i=1}^{n} X_i$, which has $\mathbb{E}[X] = n\mu$ and $\text{Var}(X) = n\sigma^2$. Let $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, which has $\mathbb{E}[\overline{X}] = \mu$ and $Var(\overline{X}) = \frac{\sigma^2}{n}$. $\overline{X}$ is called the *sample mean*. Then, as $n \to \infty$, $\overline{X}$ approaches the normal distribution $\mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$. Standardizing, this is equivalent to $Y = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ approaching $\mathcal{N}(0,1)$. Similarly, as $n \to \infty$, X approaches $\mathcal{N}(n\mu, n\sigma^2)$ and $Y' = \frac{X-n\mu}{\sigma\sqrt{n}}$ approaches $\mathcal{N}(0,1)$.

It is no surprise that $\overline{X}$ has mean μ and variance σ^2/n – this can be done with simple calculations. The importance of the CLT is that, for large n, regardless of what distribution X_i comes from, $\overline{X}$ is *approximately normally distributed with mean* μ *and variance* σ^2/n. Don't forget the continuity correction, only when $X_1, \ldots, X_n$ are discrete random variables.

2) Multivariate: Discrete to Continuous:

	Discrete	**Continuous**
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \leq x, s \leq y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(t,s)\,dsdt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dxdy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y)p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)dxdy$
Independence must have	$\forall x, y, p_{X,Y}(x,y) = p_X(x)p_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$	$\forall x, y, f_{X,Y}(x,y) = f_X(x)f_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$

Task 1 – Round-off error

Let X be the sum of 100 real numbers, and let Y be the same sum, but with each number rounded to the nearest integer before summing. If the roundoff errors are independent and uniformly distributed between -0.5 and 0.5, what is the approximate probability that $|X - Y| > 3$?

Let $X = \sum_{i=1}^{100} X_i$, and $Y = \sum_{i=1}^{100} r(X_i)$, where $r(X_i)$ is X_i rounded to the nearest integer. Then, we have

$$X - Y = \sum_{i=1}^{100} X_i - r(X_i)$$

Note that each $X_i - r(X_i)$ is simply the round off error, which is distributed as $\text{Unif}(-0.5, 0.5)$. Since $X - Y$ is the sum of 100 i.i.d. random variables with mean $\mu = 0$ and variance $\sigma^2 = \frac{1}{12}$, $X - Y \approx W \sim \mathcal{N}(0, \frac{100}{12})$ by the Central Limit Theorem. For notational convenience let $Z \sim \mathcal{N}(0,1)$

$$\begin{aligned}
\mathbb{P}(|X - Y| > 3) &\approx \mathbb{P}(|W| > 3) && \text{[CLT]} \\
&= \mathbb{P}(W > 3) + \mathbb{P}(W < -3) && \text{[No overlap between } W > 3 \text{ and } W < -3\text{]} \\
&= 2\,\mathbb{P}(W > 3) && \text{[Symmetry of normal]} \\
&= 2\,\mathbb{P}\left(\frac{W}{\sqrt{100/12}} > \frac{3}{\sqrt{100/12}}\right) \\
&\approx 2\,\mathbb{P}(Z > 1.039) && \text{[Standardize } W\text{]} \\
&= 2\,(1 - \Phi(1.039)) \approx 0.29834
\end{aligned}$$

Task 2 – Tweets

A prolific Twitter user tweets approximately 350 tweets per week. Let's assume for simplicity that the tweets are independent, and each consists of a uniformly random number of characters between 10 and 140. (Note that this is a discrete uniform distribution.) Thus, the central limit theorem (CLT) implies that the number of characters tweeted by this user is approximately normal with an appropriate mean and variance. Assuming this normal approximation is correct, estimate the probability that this user tweets between 26,000 and 27,000 characters in a particular week. (This is a case where continuity correction will make virtually no difference in the answer, but you should still use it to get into the practice!).

Let X be the total number of characters tweeted by a twitter user in a week. Let $X_i \sim Unif(10, 140)$ be the number of characters in the ith tweet (since the start of the week). Since X is the sum of 350 i.i.d. rvs with mean $\mu = 75$ and variance $\sigma^2 = 1430$, $X \approx N \sim \mathcal{N}(350 \cdot 75, 350 \cdot 1430)$. Thus,

$$\mathbb{P}(26,000 \leqslant X \leqslant 27,000) \approx \mathbb{P}(25,999.5 \leqslant N \leqslant 27,000.5)$$

Standardizing this gives the following formula

$$\begin{aligned}
\mathbb{P}(25,999.5 \leqslant N \leqslant 27,000.5) &\approx \mathbb{P}\left(-0.3541 \leqslant \frac{N - 350 \cdot 75}{\sqrt{350 \cdot 1430}} \leqslant 1.0608\right) \\
&= \mathbb{P}\left(-0.3541 \leqslant Z \leqslant 1.0608\right) \\
&= \Phi(1.0608) - \Phi(-0.3541) \\
&\approx 0.4923
\end{aligned}$$

So the probability that this user tweets between 26,000 and 27,000 characters in a particular week is approximately 0.4923.

Task 3 – Joint PMF's

Suppose X and Y have the following joint PMF:

X/Y	1	2	3
0	0	0.2	0.1
1	0.3	0	0.4

a) Identify the range of X (Ω_X), the range of Y (Ω_Y), and their joint range ($\Omega_{X,Y}$).

$\Omega_X = \{0, 1\}$, $\Omega_Y = \{1, 2, 3\}$, and $\Omega_{X,Y} = \{(0, 2), (0, 3), (1, 1), (1, 3)\}$

b) Find the marginal PMF for X, $p_X(x)$ for $x \in \Omega_X$.

$$p_X(0) = \sum_y p_{X,Y}(0, y) = 0 + 0.2 + 0.1 = 0.3$$

$$p_X(1) = 1 - p_X(0) = 0.7$$

c) Find the marginal PMF for Y, $p_Y(y)$ for $y \in \Omega_Y$.

$$p_Y(1) = \sum_x p_{X,Y}(x, 1) = 0 + 0.3 = 0.3$$

$$p_Y(2) = \sum_x p_{X,Y}(x, 2) = 0.2 + 0 = 0.2$$

$$p_Y(3) = \sum_x p_{X,Y}(x, 3) = 0.1 + 0.4 = 0.5$$

d) Are X and Y independent? Why or why not?

No, since a necessary condition is that $\Omega_{X,Y} = \Omega_X \times \Omega_Y$.

e) Find $\mathbb{E}[X^3Y]$.

Note that $X^3 = X$ since X takes values in $\{0, 1\}$.

$$\mathbb{E}[X^3Y] = \mathbb{E}[XY] = \sum_{(x,y)\in\Omega_{X,Y}} xyp_{X,Y}(x,y) = 1 \cdot 1 \cdot 0.3 + 1 \cdot 3 \cdot 0.4 = 1.5$$

Task 4 – Do You "Urn" to Learn More About Probability?

Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let $X_i = 1$ if the i-th ball selected is white and let it be equal to 0 otherwise. Give the joint probability mass function of

a) X_1, X_2

Here is one way of defining the joint pmf of X_1, X_2

$$p_{X_1,X_2}(1,1) = \mathbb{P}(X_1 = 1)\,\mathbb{P}(X_2 = 1 \mid X_1 = 1) = \frac{5}{13} \cdot \frac{4}{12} = \frac{20}{156}$$

$$p_{X_1,X_2}(1,0) = \mathbb{P}(X_1 = 1)\,\mathbb{P}(X_2 = 0 \mid X_1 = 1) = \frac{5}{13} \cdot \frac{8}{12} = \frac{40}{156}$$

$$p_{X_1,X_2}(0,1) = \mathbb{P}(X_1 = 0)\,\mathbb{P}(X_2 = 1 \mid X_1 = 0) = \frac{8}{13} \cdot \frac{5}{12} = \frac{40}{156}$$

$$p_{X_1,X_2}(0,0) = \mathbb{P}(X_1 = 0)\,\mathbb{P}(X_2 = 0 \mid X_1 = 0) = \frac{8}{13} \cdot \frac{7}{12} = \frac{56}{156}$$

b) X_1, X_2, X_3

Instead of listing out all the individual probabilities, we could write a more compact formula for the pmf. In this problem, the denominator is always $P(13, k)$, where k is the number of random variables in the joint pmf. And the numerator is $P(5, i)$ times $P(8, j)$ where i and j are the number of 1s and 0s, respectively.

If we wish to compute $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$, then the number of 1s (i.e., white balls) is $x_1+x_2+x_3$, and the number of 0s (i.e., red balls) is $(1 - x_1) + (1 - x_2) + (1 - x_3)$. Then, we can write the pmf as follows:

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{10!}{13!} \cdot \frac{5!}{(5 - x_1 - x_2 - x_3)!} \cdot \frac{8!}{(5 + x_1 + x_2 + x_3)!}$$

Task 5 – Continuous joint density

The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

and the joint density of W and V is given by

$$f_{W,V}(w,v) = \begin{cases} 2 & 0 < w < v, 0 < v < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent? Are W and V independent?

For two random variables X, Y to be independent, we must have $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all $x \in \Omega_X,\ y \in \Omega_Y$. Let's start with X and Y by finding their marginal PDFs. By definition, and using the fact that the joint PDF is 0 outside of $y > 0$, we get:

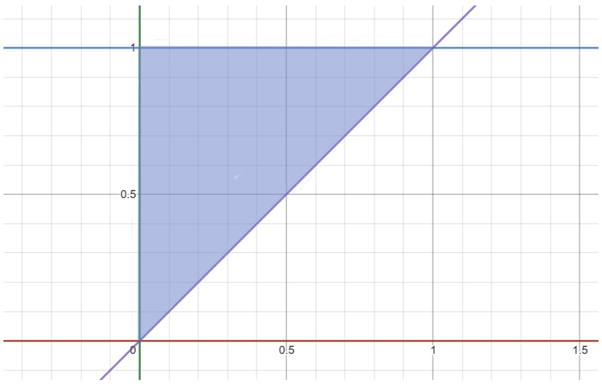
$$f_X(x) = \int_0^{\infty} xe^{-(x+y)}dy = e^{-x}x$$

We do the same to get the PDF of Y, again over the range $x > 0$:

$$f_Y(y) = \int_0^{\infty} xe^{-(x+y)}dx = e^{-y}$$

Since $e^{-x}x \cdot e^{-y} = xe^{-x-y} = xe^{-(x+y)}$ for all $x, y > 0$, X and Y are independent.

We can see that W and V are not independent simply by observing that $\Omega_W = (0,1)$ and $\Omega_V = (0,1)$, but $\Omega_{W,V}$ is not equal to their Cartesian product. Specifically, looking at their range of $f_{W,V}(w,v)$. Graphing it with w as the "x-axis" and v as the "y-axis", we see that :



The shaded area is where the joint pdf is strictly positive. Looking at it, we can see that it is not rectangular, and therefore it is not the case that $\Omega_{W,V} = \Omega_W \times \Omega_V$. Remember, the joint range being the Cartesian product of the marginal ranges is not sufficient for independence, but it is *necessary*. Therefore, this is enough to show that they are not independent.

Task 6 – Trapped Miner

A miner is trapped in a mine containing 3 doors.

- D_1: The 1^{st} door leads to a tunnel that will take him to safety after 3 hours.
- D_2: The 2^{nd} door leads to a tunnel that returns him to the mine after 5 hours.
- D_3: The 3^{rd} door leads to a tunnel that returns him to the mine after a number of hours that is Binomial with parameters $(12, \frac{1}{3})$.

At all times, he is equally likely to choose any one of the doors. What is the expected number of hours for this miner to reach safety?

Let T = number of hours for the miner to reach safety. (T is a random variable)
Let D_i be the event the i^{th} door is chosen. $i \in \{1, 2, 3\}$. Finally, let T_3 be the time it takes to return to the mine in the third case only (a random variable). Note that the expectation of T_3 is $12 * \frac{1}{3}$ because it is binomially distributed with parameters $n = 12, p = \frac{1}{3}$. By Law of Total Expectation, linearity of expectation, and by applying the conditional expectations given by the problem statement:

$$\begin{aligned}
\mathbb{E}[T] &= \mathbb{E}[T|D_1] \cdot \mathbb{P}(D_1) + \mathbb{E}[T|D_2] \cdot \mathbb{P}(D_2) + \mathbb{E}[T|D_3] \cdot \mathbb{P}(D_3) \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (\mathbb{E}[T_3 + T]) \cdot \frac{1}{3} \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (\mathbb{E}[T_3] + \mathbb{E}[T]) \cdot \frac{1}{3} \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (4 + \mathbb{E}[T]) \cdot \frac{1}{3}
\end{aligned}$$

Solving this equation for $\mathbb{E}[T]$, we get

$$\mathbb{E}[T] = 12$$

Therefore, the expected number of hours for this miner to reach safety is 12.

Task 7 – Lemonade Stand

Suppose I run a lemonade stand, which costs me \$100 a day to operate. I sell a drink of lemonade for \$20. Every person who walks by my stand either buys a drink or doesn't (no one buys more than one). If it is raining, n_1 people walk by my stand, and each buys a drink independently with probability p_1. If it isn't raining, n_2 people walk by my stand, and each buys a drink independently with probability p_2. It rains each day with probability p_3, independently of every other day. Let X be my profit over the next week. In terms of n_1, n_2, p_1, p_2 and p_3, what is $\mathbb{E}[X]$?

Let R be the event it rains. Let X_i be how many drinks I sell on day i for $i = 1, \ldots, 7$. We are interested in $X = \sum_{i=1}^{7} (20X_i - 100)$. We have $X_i|R \sim$ Binomial(n_1, p_1), so $\mathbb{E}[X_i|R] = n_1p_1$. Similarly, $X_i|R^C \sim$ Binomial(n_2, p_2), so $\mathbb{E}[X_i|R^C] = n_2p_2$. By the law of total expectation,

$$\mu = \mathbb{E}[X_i] = \mathbb{E}[X_i|R]\mathbb{P}(R) + \mathbb{E}[X_i|R^C]\mathbb{P}\left(R^C\right) = n_1p_1p_3 + n_2p_2(1 - p_3)$$

Hence, by linearity of expectation,

$$\begin{aligned}
\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{7} (20X_i - 100)] = 20 \sum_{i=1}^{7} \mathbb{E}[X_i] - 700 = 140\mu - 700 \\
= 140 \cdot (n_1p_1p_3 + n_2p_2(1 - p_3)) - 700.
\end{aligned}$$