Quiz Section 8

Review

1) Markov's Inequality: Let X be a non-negative random variable, and $\alpha > 0$. Then, $\mathbb{P}(X \geqslant \alpha) \leqslant \frac{\mathbb{E}[X]}{\alpha}$.

2) Chebyshev's Inequality: Suppose Y is a random variable with $\mathbb{E}[Y] = \mu$ and $\mathrm{Var}(Y) = \sigma^2$. Then, for any $\alpha > 0$, $\mathbb{P}(|Y - \mu| \geqslant \alpha) \leqslant \frac{\sigma^2}{\alpha^2}$.

3) Chernoff Bound: Suppose $X = X_1 + \cdots + X_n$ where the X_i are independent and in $[0, 1]$. Let $\mu = \mathbb{E}[X]$. Then, for any $0 < \delta \leqslant 1$, $\mathbb{P}(|X - \mu| \geqslant \delta\mu) \leqslant e^{-\ \delta^2\mu/4}$ and for any $\delta > 0$, $\mathbb{P}(X - \mu \geqslant \delta\mu) \leqslant e^{-\delta^2\mu/4}$.

4) Multivariate: Discrete to Continuous:

	Discrete	**Continuous**
Joint PMF/PDF	$p_{X,Y}(x, y) = \mathbb{P}(X = x, Y = y)$	$f_{X,Y}(x, y) \neq \mathbb{P}(X = x, Y = y)$
Joint range/support $\Omega_{X,Y}$	$\{(x, y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x, y) > 0\}$	$\{(x, y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x, y) > 0\}$
Joint CDF	$F_{X,Y}(x, y) = \sum_{t \leqslant x, s \leqslant y} p_{X,Y}(t, s)$	$F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t, s)\, ds dt$
Normalization	$\sum_{x,y} p_{X,Y}(x, y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x, y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$
Expectation	$\mathbb{E}[g(X, Y)] = \sum_{x,y} g(x, y) p_{X,Y}(x, y)$	$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx dy$
Independence must have	$\forall x, y, p_{X,Y}(x, y) = p_X(x) p_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$	$\forall x, y, f_{X,Y}(x, y) = f_X(x) f_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$
Conditional PMF/PDF	$p_{X\mid Y}(x\mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\mid Y}(x\mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$\mathbb{E}[X\mid Y = y] = \sum_x x \cdot p_{X\mid Y}(x\mid y)$	$\mathbb{E}[X\mid Y = y] = \int_{-\infty}^{\infty} x f_{X\mid Y}(x\mid y) dx$

Task 1 – A Dysfunctional Family

Rick and his grandson Morty are set to meet at a certain time. Since their relationship is a little strained, neither of them wants to be there on time. Let $X \sim Unif(0, 10)$ be the amount of minutes Morty is going to be late. Rick has cameras around the meeting spot and will observe Morty's arrival time $X = x$. Then, he will arrive at the meeting spot $Unif(x, 5x)$ minutes late. Let Y be the random variable indicating how late Rick will be.

a) Using the above definitions determine f_X, $f_{Y|X}$, and f_{XY}. (You will want to determine f_{YX} and use it to determine f_{XY}.).

b) Compute $\mathbb{E}[Y]$.

Task 2 – Tail bounds

Suppose $X \sim \text{Binomial}(6, 0.4)$. We will bound $\mathbb{P}(X \geqslant 4)$ using the tail bounds we've learned, and compare this to the true result.

a) Give an upper bound for this probability using Markov's inequality. Why can we use Markov's inequality?

b) Give an upper bound for this probability using Chebyshev's inequality. You may have to rearrange algebraically and it may result in a weaker bound.

c) Give an upper bound for this probability using the Chernoff bound.

d) Give the exact probability.

Task 3 – Exponential Tail Bounds

Let $X \sim \mathsf{Exp}(\lambda)$ and $k > 1/\lambda$. Recall that $\mathbb{E}[X] = \frac{1}{\lambda}$ and $\operatorname{Var}(X) = \frac{1}{\lambda^2}$.

a) Use Markov's inequality to bound $\mathsf{P}(X \geqslant k)$.

b) Use Chebyshev's inequality to bound $\mathsf{P}(X \geqslant k)$.

c) What is the exact formula for $\mathsf{P}(X \geqslant k)$?

d) For $\lambda k \geqslant 3$, how do the bounds given in parts (a), (b), and (c) compare?

Task 4 – How many samples?

Let $X = X_1 + \ldots X_n$ be the sum of n independent $Poisson(\lambda)$ random variables. Recall that the Poisson distribution has expectation and variance both equal to λ and has the summation property that X is a $Poisson(n\lambda)$ random variable.

a) How large a value of n would Chebyshev's inequality need to guarantee that $\mathbb{P}(X \leqslant \mathbb{E}[X]/2) \leqslant 0.01$?

b) How large a value of n would Markov's inequality need to guarantee that $\mathbb{P}(X \leqslant \mathbb{E}[X]/2) \leqslant 0.01$?