CSE 312

# Foundations of Computing II

## Lecture 8: Linearity of Expectation

## Last Class:

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation

## Today:

- More Expectation Examples
- Linearity of Expectation
- Indicator Random Variables

**Kandinsky**

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## Review Random Variables

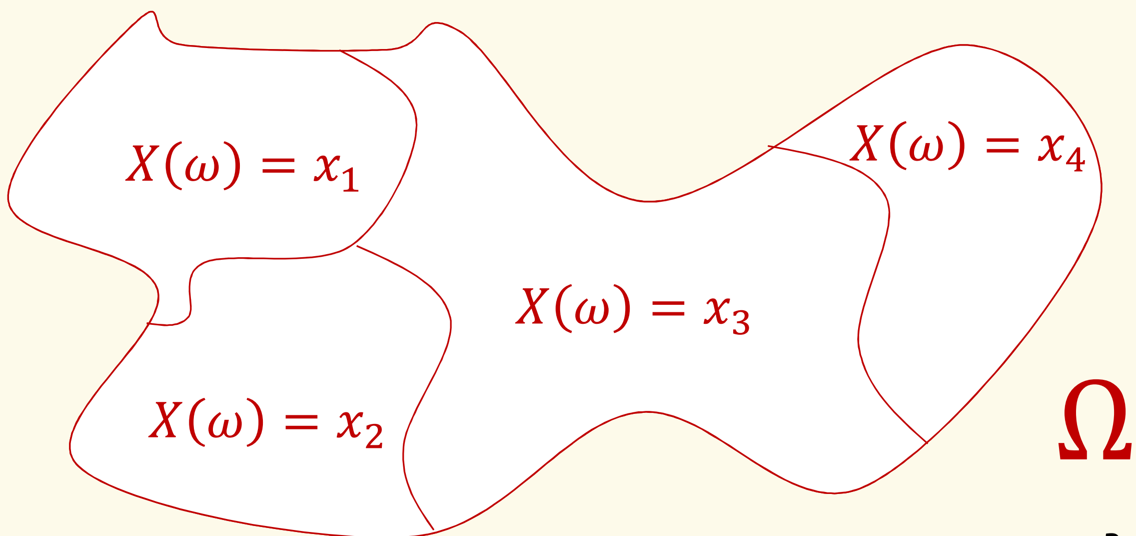
**Definition.** A **random variable (RV)** for a probability space $(\Omega, P)$ is a function $X: \Omega \to \mathbb{R}$.

The set of values that $X$ can take on is its *range/support*: $X(\Omega)$ or $\Omega_X$

$\{X = x_i\} = \{\omega \in \Omega \mid X(\omega) = x_i\}$

Random variables **partition** the sample space.

$\Sigma_{x \in X(\Omega)} P(X = x) = 1$



# Review PMF and CDF

**Definitions:**

For a RV $X: \Omega \to \mathbb{R}$, the probability mass function (pmf) of $X$ specifies, for any real number $x$, the probability that $X = x$

$$p_X(x) = P(X = x) = P(\{\omega \in \Omega \mid X(\omega) = x\})$$

$$\boxed{\sum_{x \in \Omega_X} p_X(x) = 1}$$

For a RV $X: \Omega \to \mathbb{R}$, the cumulative distribution function (cdf) of $X$ specifies, for any real number $x$, the probability that $X \leq x$

$$F_X(x) = P(X \leq x)$$

## Review Expected Value of a Random Variable

**Definition.** Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of $X$ is

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

or equivalently

$$\mathbb{E}[X] = \sum_{x \in \mathrm{X}(\Omega)} x \cdot P(X = x) = \sum_{x \in \Omega_X} x \cdot p_X(x)$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

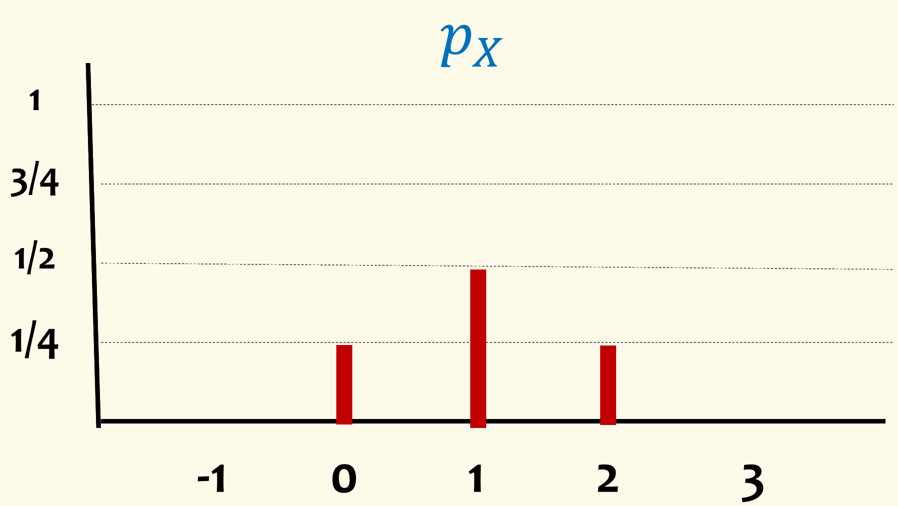
# Expectation

**Example.** Two fair coin flips

$\Omega = \{\mathrm{TT}, \mathrm{HT}, \mathrm{TH}, \mathrm{HH}\}$

$X =$ number of heads

What is $\mathbb{E}[X]$?




$$\mathbb{E}[X] = 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = \frac{1}{2} + \frac{1}{2} = 1$$

0

## Another Interpretation

*"If $X$ is how much you win playing the game in one round. How much would you expect to win, on average, per game, when repeatedly playing?"*

**Answer:** $\mathbb{E}[X]$

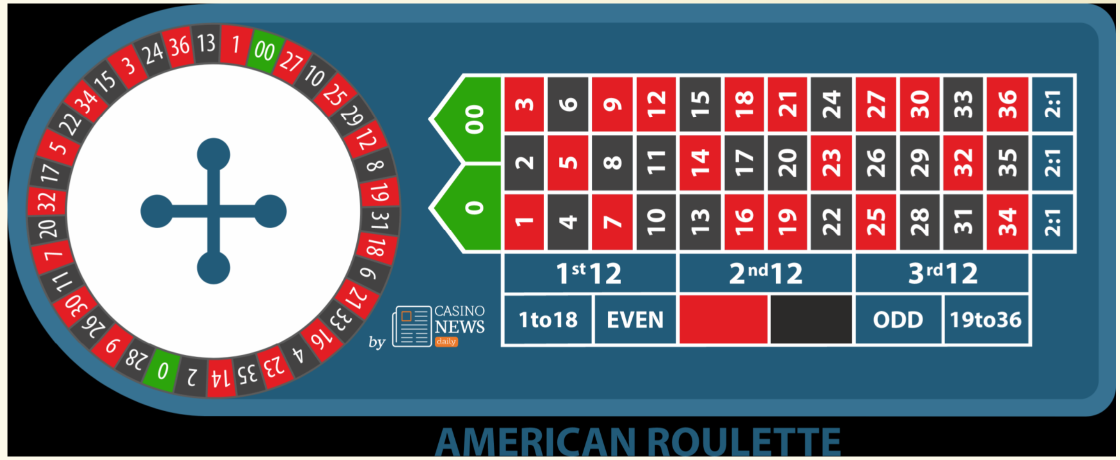
# Roulette (USA)

$\Omega$:

Numbers 1-36

- 18 Red
- 18 Black

Green 0 and 00




Note 0 and 00 are not EVEN

RVs for gains from some bets:

RV RED: If Red number turns up $+1$, if Black number, 0, or 00 turns up $-1$

$$\mathbb{E}[\text{RED}] = (+1)\cdot\frac{18}{38} + (-1)\cdot\frac{20}{38} = -\frac{2}{38} \approx -5.26\%$$

RV $1^{st}12$: If number 1-12 turns up $+2$, if number 13-36, 0, or 00 turns up $-1$

$$\mathbb{E}[1^{st}12] = (+2)\cdot\frac{12}{38} + (-1)\cdot\frac{26}{38} = -\frac{2}{38} \approx -5.26\%$$

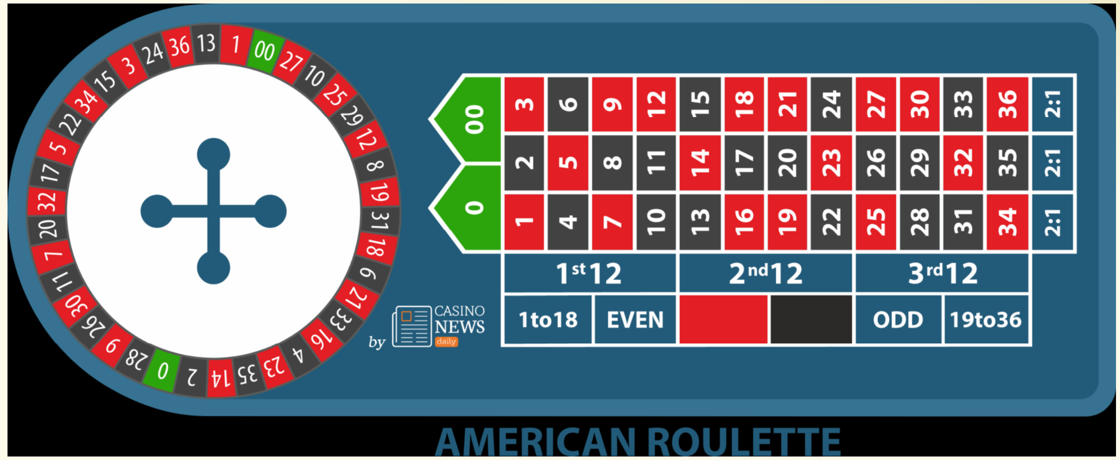
# Roulette (USA)

$\Omega$:

Numbers 1-36

- 18 Red
- 18 Black

Green 0 and 00




Note 0 and 00 are not EVEN

An even worse bet:

RV BASKET: If 0, 00, 1, 2, or 3 turns up +6 otherwise $-1$

$$\mathbb{E}[\text{BASKET}] = (+6) \cdot \frac{5}{38} + (-1) \cdot \frac{33}{38} = -\frac{3}{38} \approx -7.89\%$$

## Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let $X$ be the number of students who get their own HW

| $\mathbf{Pr}(\boldsymbol{\omega})$ | $\boldsymbol{\omega}$ | $\boldsymbol{X}(\boldsymbol{\omega})$ |
|---|---|---|
| $1/6$ | $1, 2, 3$ | $3$ |
| $1/6$ | $1, 3, 2$ | $1$ |
| $1/6$ | $2, 1, 3$ | $1$ |
| $1/6$ | $2, 3, 1$ | $0$ |
| $1/6$ | $3, 1, 2$ | $0$ |
| $1/6$ | $3, 2, 1$ | $1$ |

$$\mathbb{E}[X] = 3 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6}$$

$$= 6 \cdot \frac{1}{6} = 1$$

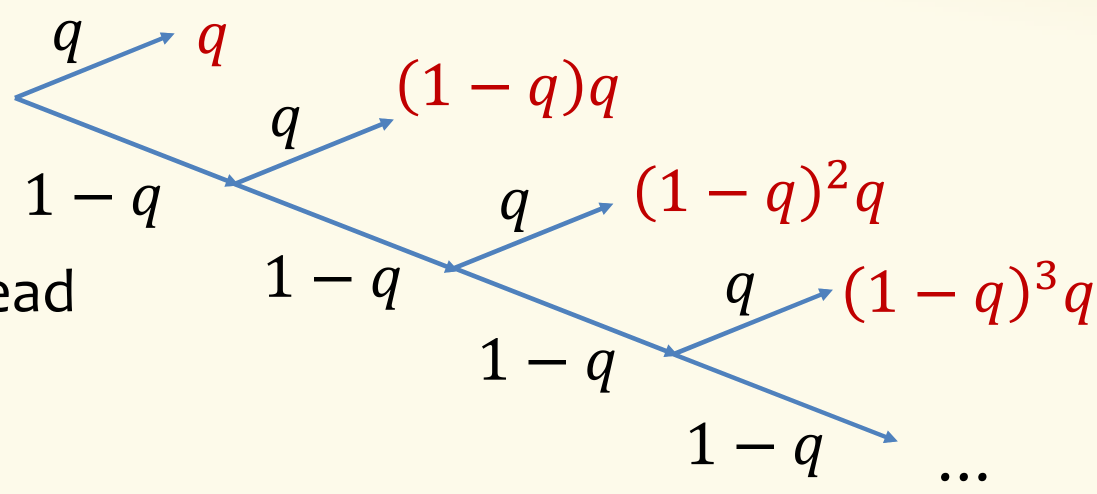
# Example – Flipping a biased coin until you see heads

- Biased coin:

  $P(H) = q > 0$

  $P(T) = 1 - q$

- $Z$ = # of coin flips until first head




$$P(Z = i) = q\,(1-q)^{i-1}$$

$$\mathbb{E}[Z] = \sum_{i=1}^{\infty} i \cdot P(Z = i) = \sum_{i=1}^{\infty} i \cdot q(1-q)^{i-1}$$

Converges, so $\mathbb{E}[Z]$ is finite

Can calculate this directly but…

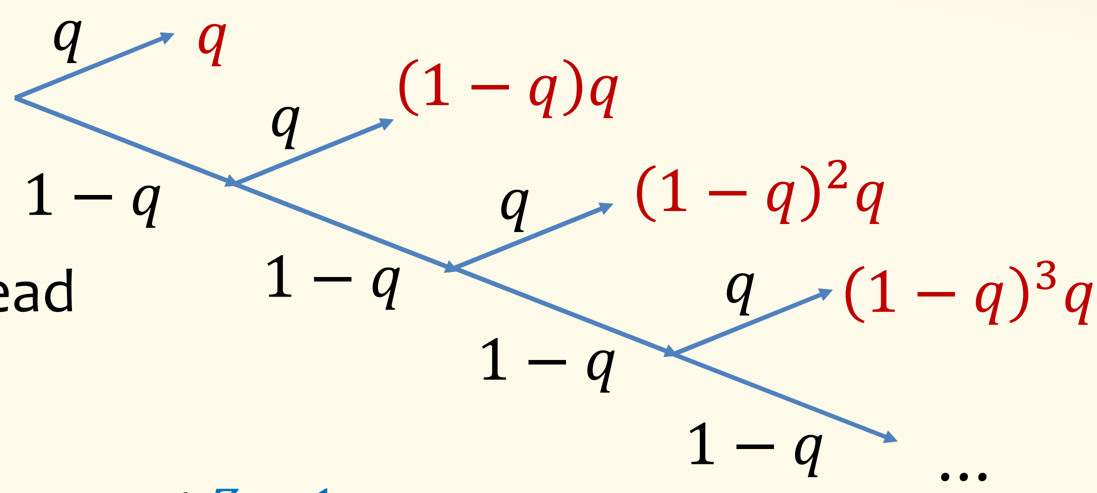
# Example – Flipping a biased coin until you see heads

- Biased coin:

  $P(H) = q > 0$

  $P(T) = 1 - q$

- $Z$ = # of coin flips until first head




**Another view:** If you get heads first try you get $Z = 1$;
If you get tails you have used one try and have the same experiment left

$$\mathbb{E}[Z] = q + (1-q)(1 + \mathbb{E}(Z))$$

Solving gives $\quad q \cdot \mathbb{E}[Z] = q + (1-q) = 1 \quad$ Implies $\boxed{\mathbb{E}[Z] = 1/q}$

# Expected Value of $X$ = # of heads

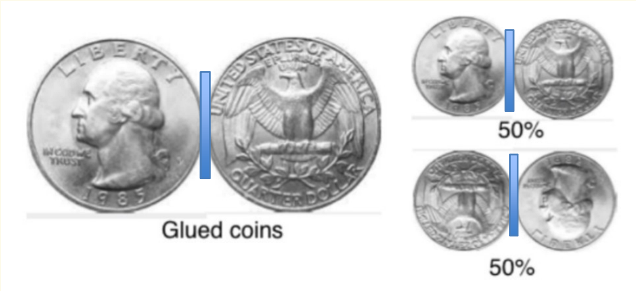
Each coin shows up heads half the time.

Two fair coins

$P(HT) = P(TH) = 0.25$
$P(HH) = P(TT) = 0.25$

$$\mathbb{E}(X) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

Glued coins




$P(HT) = P(TH) = 0.5$
$P(HH) = P(TT) = 0$

$$\mathbb{E}(X) = 1 \cdot 1 = 1$$

Attached coins


$P(HH) = P(TT) = 0.4$
$P(HT) = P(TH) = 0.1$

$$\mathbb{E}(X) = 1 \cdot 0.2 + 2 \cdot 0.4 = 1$$

## Linearity of Expectation

**Theorem.** For any two random variables $X$ and $Y$

*($X, Y$ do not need to be independent)*

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

Or, more generally: For any random variables $X_1, \dots, X_n$,

$$\mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

**Because:** $\mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[(X_1 + \cdots + X_{n-1}) + X_n]$

$$= \mathbb{E}[X_1 + \cdots + X_{n-1}] + \mathbb{E}[X_n] = \cdots$$

# Linearity of Expectation – Proof

**Theorem.** For any two random variables $X$ and $Y$
*($X, Y$ do not need to be independent)*

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

$$\begin{aligned}
\mathbb{E}[X+Y] &= \sum_\omega P(\omega)(X(\omega) + Y(\omega)) \\
&= \sum_\omega P(\omega)X(\omega) + \sum_\omega P(\omega)Y(\omega) \\
&= \mathbb{E}[X] + \mathbb{E}[Y]
\end{aligned}$$

## Example – Coin Tosses

We flip $n$ coins, each one heads with probability $p$

$Z$ is the number of heads, what is $\mathbb{E}(Z)$?

# Example – Coin Tosses – The brute force method

We flip $n$ coins, each one heads with probability $p$,

$Z$ is the number of heads, what is $\mathbb{E}[Z]$?

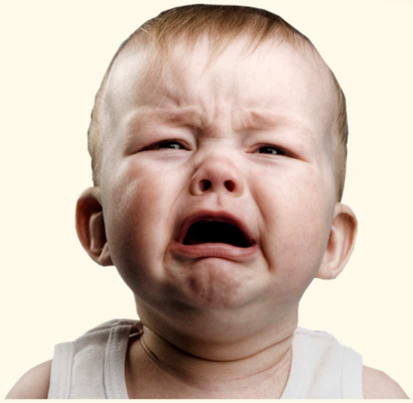
$$\mathbb{E}[Z] = \sum_{k=0}^{n} k \cdot P(Z = k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} k \cdot \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} = \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!} p^k (1-p)^{(n-1)-k}$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k} = np\big(p + (1-p)\big)^{n-1} = np \cdot 1 = np$$



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Can we solve it more elegantly, please?

## Computing complicated expectations

Often boils down to the following three steps:

- Decompose: Finding the right way to decompose the random variable into sum of simple random variables
$$X = X_1 + \cdots + X_n$$
- LOE: Apply linearity of expectation.
$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$
- Conquer: Compute the expectation of each $X_i$

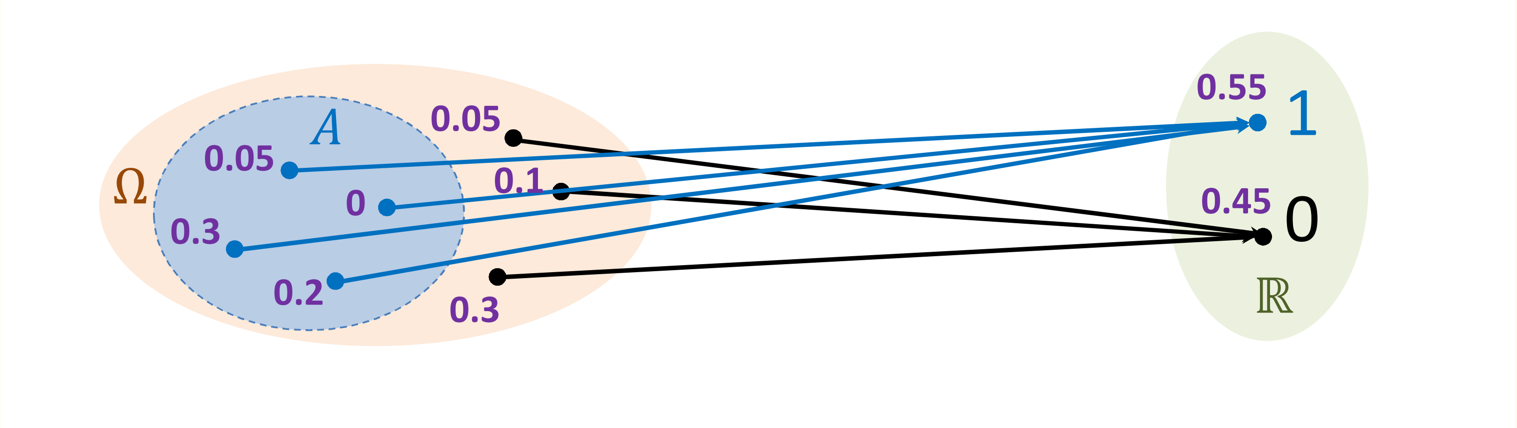
Often, $X_i$ are indicator (0/1) random variables.

# Indicator random variables

For any event $A$, can define the indicator random variable $X_A$ for $A$

$$X_A = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{if event } A \text{ does not occur} \end{cases}$$

$$\boxed{\begin{aligned} P(X_A = 1) &= P(A) \\ P(X_A = 0) &= 1 - P(A) \end{aligned}}$$



## Example – Coin Tosses

We flip $n$ coins, each one heads with probability $p$

$Z$ is the number of heads, what is $\mathbb{E}[Z]$?

- $X_i = \begin{cases} 1, & i^{\text{th}} \text{ coin flip is heads} \\ 0, & i^{\text{th}} \text{ coin flip is tails.} \end{cases}$

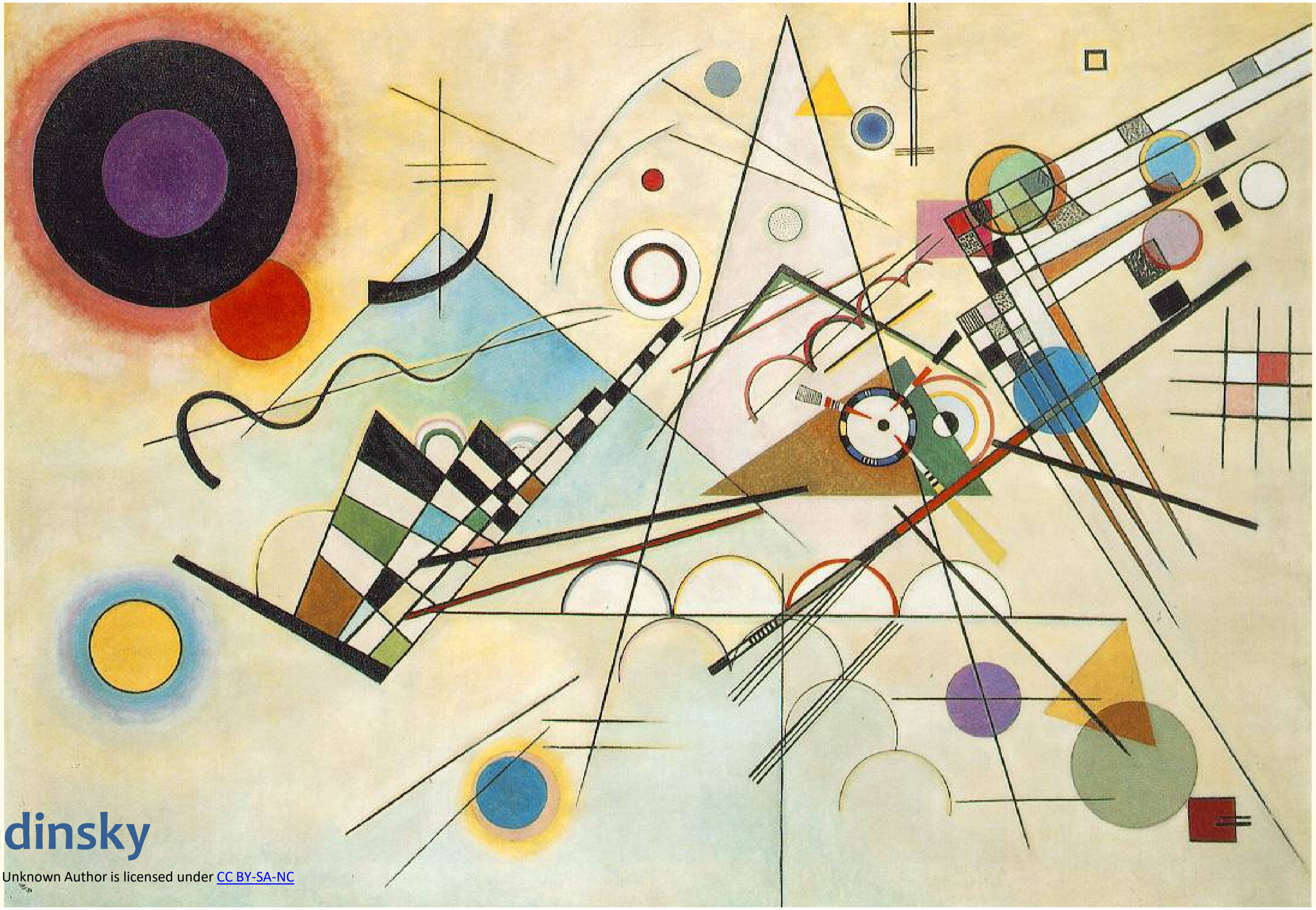
**Fact.** $Z = X_1 + \cdots + X_n$

**Linearity of Expectation:**

$$\mathbb{E}[Z] = \mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] \quad = n \cdot p$$

$P(X_i = 1) = p$
$P(X_i = 0) = 1 - p$

$$\mathbb{E}[X_i] = p \cdot 1 + (1 - p) \cdot 0 = p$$



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## Example: Returning Homeworks

- Class with $n$ students, randomly hand back homeworks. All permutations equally likely.
- Let $X$ be the number of students who get their own HW

What is $\mathbb{E}[X]$? Use linearity of expectation!

| $\mathbf{Pr(\omega)}$ | $\boldsymbol{\omega}$ | $\boldsymbol{X(\omega)}$ |
|---|---|---|
| $1/6$ | $1, 2, 3$ | $3$ |
| $1/6$ | $1, 3, 2$ | $1$ |
| $1/6$ | $2, 1, 3$ | $1$ |
| $1/6$ | $2, 3, 1$ | $0$ |
| $1/6$ | $3, 1, 2$ | $0$ |
| $1/6$ | $3, 2, 1$ | $1$ |

Decompose: What is $X_i$?

$X_i = 1$ iff $i^{th}$ student gets own HW back

LOE: $\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n]$

Conquer: What is $\mathbb{E}[X_i]$? A. $\frac{1}{n}$ B. $\frac{1}{n-1}$ C. $\frac{1}{2}$

Poll: **pollev.com/paulbeame028**

## Pairs with the same birthday

- In a class of $m$ students, on average how many pairs of people have the same birthday (assuming 365 equally likely birthdays)?

Decompose: Indicator events involve **pairs** of students $(i, j)$ for $i \neq j$

$X_{ij} = 1$ iff students $i$ and $j$ have the same birthday

LOE: $\binom{m}{2}$ indicator variables $X_{ij}$

Conquer: $\mathbb{E}[X_{ij}] = \frac{1}{365}$ so total expectation is $\frac{\binom{m}{2}}{365} = \frac{m(m-1)}{730}$ pairs

# Linearity of Expectation – Even stronger

**Theorem.** For any random variables $X_1, \dots, X_n$, and real numbers $a_1, \dots, a_n \in \mathbb{R}$,

$$\mathbb{E}[a_1 X_1 + \cdots + a_n X_n] = a_1 \mathbb{E}[X_1] + \cdots + a_n \mathbb{E}[X_n].$$

Very important: In general, we do <u>not</u> have $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

## Linearity is special!

In general $\mathbb{E}[g(X)] \neq g\big(\mathbb{E}(X)\big)$

E.g., $X = \begin{cases} +1 \text{ with prob } 1/2 \\ -1 \text{ with prob } 1/2 \end{cases}$

Then: $\mathbb{E}[X^2] \neq \mathbb{E}[X]^2$

How DO we compute $\mathbb{E}[g(X)]$?

# Expected Value of $g(X)$

**Definition.** Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of $g(X)$ is

$$\mathbb{E}[g(X)] = \sum_{\omega \in \Omega} g\big(X(\omega)\big) \cdot P(\omega)$$

or equivalently

$$\mathbb{E}[g(X)] = \sum_{x \in \mathrm{X}(\Omega)} g(x) \cdot P(X = x) = \sum_{x \in \Omega_X} g(x) \cdot p_X(x)$$

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