CSE 312

# Foundations of Computing II

## Lecture 12: Zoo of Discrete RVS part II
## Poisson Distribution

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# Announcements

- Midterm info is posted   Nov 4
  - Q&A session next Tuesday 4pm on Zoom
  - Practice midterm + other practice materials posted this Wednesday

# Zoo of Random Variables

### $X \sim \text{Unif}(a, b)$

$$P(X = k) = \frac{1}{b - a + 1}$$

$$E[X] = \frac{a + b}{2}$$

$$\text{Var}(X) = \frac{(b - a)(b - a + 2)}{12}$$

### $X \sim \text{Ber}(p)$

$$P(X = 1) = p, P(X = 0) = 1 - p$$

$$E[X] = p$$

$$\text{Var}(X) = p(1 - p)$$

### $X \sim \text{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$E[X] = np$$

$$\text{Var}(X) = np(1 - p)$$

### $X \sim \text{Geo}(p)$

$$P(X = k) = (1 - p)^{k-1} p$$

$$E[X] = \frac{1}{p}$$

$$\text{Var}(X) = \frac{1 - p}{p^2}$$

### $X \sim \text{NegBin}(r, p)$

$$P(X = k) = \binom{k - 1}{r - 1} p^r (1 - p)^{k-r}$$

$$E[X] = \frac{r}{p}$$

$$\text{Var}(X) = \frac{r(1 - p)}{p^2}$$

### $X \sim \text{HypGeo}(N, K, n)$

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

$$E[X] = n\frac{K}{N}$$

$$\text{Var}(X) = n\frac{K(N - K)(N - n)}{N^2(N - 1)}$$

# Agenda

- Zoo of Discrete RVs
  - Uniform Random Variables, Part I
  - Bernoulli Random Variables, Part I
  - Binomial Random Variables, Part I
  - Geometric Random Variables
  - Negative Binomial Random Variables
  - Hypergeometric Random Variables
  - Poisson Distribution
    - Approximate Binomial distribution using Poisson distribution
  - Applications

## Geometric Random Variables

A discrete random variable $X$ that models the number of independent trials $Y_i \sim \text{Ber}(p)$ ~~before~~ until seeing the first success. (includes success trial)

$X$ is called a Geometric random variable with parameter $p$.

**Notation:** $X \sim \text{Geo}(p)$

**PMF:**

**Expectation:** $\frac{1}{p}$

**Variance:**

Examples:

- \# of coin flips until first head
- \# of random guesses on MC questions until you get one right
- \# of random guesses at a password until you hit it

# Geometric Random Variables

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$X$ is called a Geometric random variable with parameter $p$.

**Notation:** $X \sim \text{Geo}(p)$

**PMF:** $P(X = k) = (1 - p)^{k-1}p$

**Expectation:** $\mathbb{E}[X] = \frac{1}{p}$

**Variance:** $\text{Var}(X) = \frac{1-p}{p^2}$

Examples:

- # of coin flips until first head
- # of random guesses on MC questions until you get one right
- # of random guesses at a password until you hit it

# Example: Music Lessons

(1st note corr & 2nd note & .. 1000th note corr)

Your music teacher requires you to play a 1000 note song without mistake. You have been practicing, so you have a probability of 0.999 of getting each note correct (independent of the others). If you mess up a single note in the song, you must start over and play from the beginning. Let $X$ be the number of times you have to play the song from the start. What is $\mathbb{E}[X]$?

$$\text{Prob}[\text{1 playing is good}] = (0.999)^{1000}$$

$$p \leftarrow = \left(1 - \frac{1}{1000}\right)^{1000} \approx e^{-1}$$

Geo(p)

$$\mathbb{E}[\#\text{ playings}] = \frac{1}{\left(1-\frac{1}{1000}\right)^{1000}} \approx e$$

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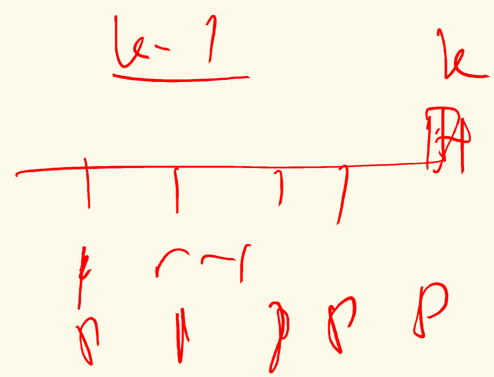
## Negative Binomial Random Variables

A discrete random variable $X$ that models the number of independent trials $Y_i \sim \text{Ber}(p)$ before seeing the $r^{th}$ success.
Equivalently, $X = \sum_{i=1}^{r} Z_i$ where $Z_i \sim \text{Geo}(p)$.
$X$ is called a Negative Binomial random variable with parameters $r, p$.




**Notation:** $X \sim \text{NegBin}(r, p)$

**PMF:** $P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$

previous successes

**Expectation:** $\mathbb{E}[X] = \frac{r}{p}$

**Variance:** $\text{Var}(X) = \frac{r(1-p)}{p^2}$

## Hypergeometric Random Variables

$\frac{K}{N}$

A discrete random variable $X$ that models the number of successes in $n$ draws (without replacement) from $N$ items that contain $K$ successes in total. $X$ is called a Hypergeometric RV with parameters $N, K, n$.

**Notation:** $X \sim \text{HypGeo}(N, K, n)$

**PMF:** $P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$

**Expectation:** $\mathbb{E}[X] = n\frac{K}{N}$

**Variance:** $\text{Var}(X) = n\frac{K(N-K)(N-n)}{N^2(N-1)}$

# Hope you enjoyed the zoo!

**$X \sim \mathrm{Unif}(a, b)$**

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**$X \sim \mathrm{Ber}(p)$**

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## Preview: Poisson

Model: # events that occur in an hour

- Expect to see 3 events per hour (but will be random)
- The expected number of events in $t$ hours, is $3t$
- Occurrence of events on disjoint time intervals is independent

## Example – Modelling car arrivals at an intersection

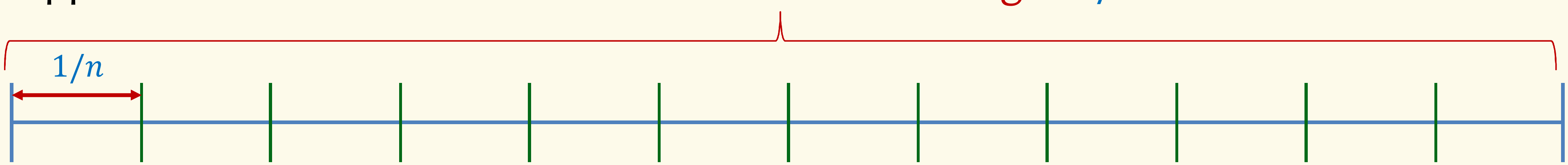
$X$ = # of cars passing through a light in 1 hour

## Example – Model the process of cars passing through a light in 1 hour

$X$ = # cars passing through a light in $1$ hour. $\mathbb{E}[X] = 3$

Assume: Occurrence of events on disjoint time intervals is independent

Approximation idea: Divide hour into $n$ intervals of length $1/n$




This gives us $n$ independent intervals

Assume at most one car per interval

$p =$ probability car arrives in an interval

Bin(n, p)

What should $p$ be?

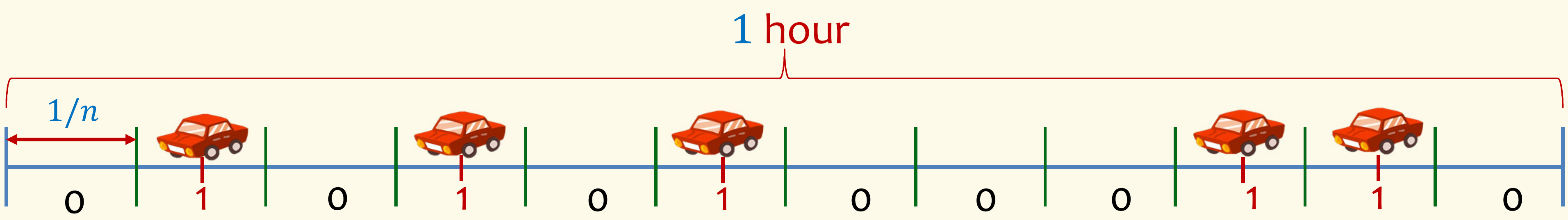
**pollev.com/paulbeame028**

A. $3/n$
B. $3n$
C. $3$
D. $3/60$

## Example – Model the process of cars passing through a light in 1 hour

$X$ = # cars passing through a light in $1$ hour. Disjoint time intervals are independent.

Know: $\mathbb{E}[X] = \lambda$ for some given $\lambda > 0$




**Discrete version:** $n$ intervals, each of length $1/n$ .

In each interval, there is a car with probability $p = \lambda/n$ (assume $\leq 1$ car can pass by)

**Each interval is Bernoulli:** $X_i = 1$ if car in $i^{\text{th}}$ interval ($0$ otherwise). $P(X_i = 1) = \lambda/n$

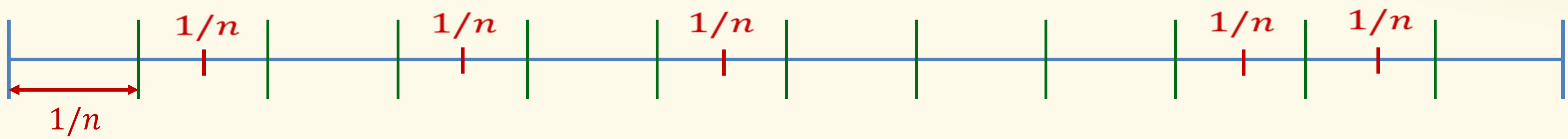
$X = \sum_{i=1}^{n} X_i$ $\quad X \sim \text{Bin}(n, p)$ $\quad P(X = i) = \binom{n}{i} \left(\frac{\lambda}{n}\right)^{i} \left(1 - \frac{\lambda}{n}\right)^{n-i}$

indeed! $\mathbb{E}[X] = pn = \lambda$

$p^i \; (1-p)^{n-i}$

# Don't like discretization

$X$ is binomial $P(X=i) = \binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i}$




**We want now** $n \to \infty$

$\frac{n(n-1)\cdots(n-i+1)}{n^i}$ $e^{-\lambda}$

$$P(X=i) = \binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i} = \underbrace{\frac{n!}{(n-i)!\,n^i}}_{\to 1}\frac{\lambda^i}{i!}\underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{\to e^{-\lambda}}\underbrace{\left(1-\frac{\lambda}{n}\right)^{-i}}_{\to 1}$$ $n \to \infty$

$$\to P(X=i) = e^{-\lambda}\cdot\frac{\lambda^i}{i!}$$

## Poisson Distribution

Siméon Denis Poisson
1781-1840

- Suppose "events" happen, independently, at an *average* rate of $\lambda$ per unit time.
- Let $X$ be the *actual* number of events happening in a given time unit. Then $X$ is a *Poisson* r.v. *with parameter* $\lambda$ (denoted $X \sim \text{Poi}(\lambda)$) and has distribution (PMF):

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

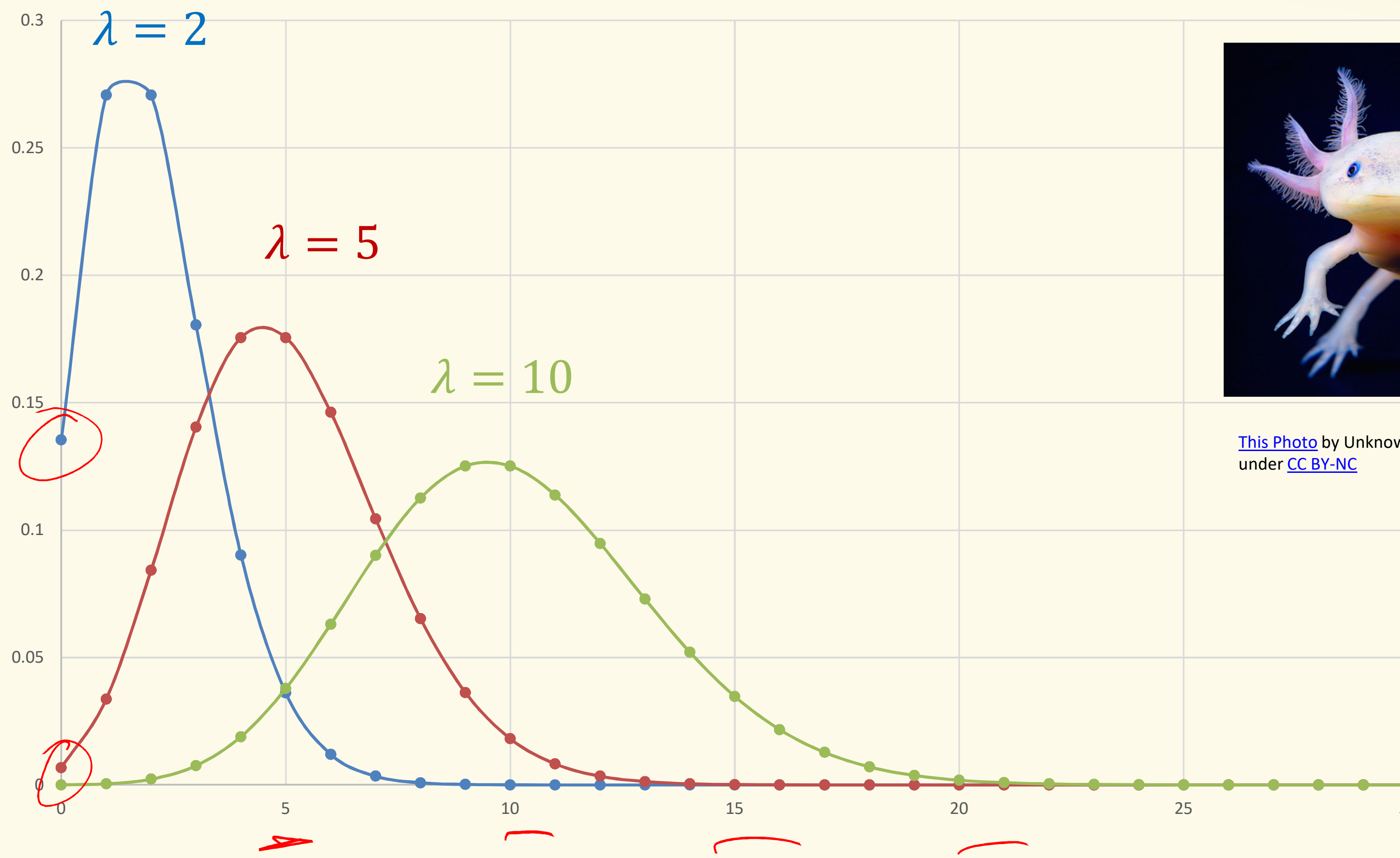
Several examples of "Poisson processes":

- # of cars passing through a traffic light in 1 hour
- # of requests to web servers in an hour
- # of photons hitting a light detector in a given interval
- # of patients arriving to ER within an hour

Assume fixed average rate

# Probability Mass Function

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$




## Validity of Distribution

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

We first want to verify that Poisson probabilities sum up to 1.

$$\sum_{i=0}^{\infty} P(X = i) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1$$

**Fact (Taylor series expansion):**

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

## Expectation

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

**Theorem.** If $X$ is a Poisson RV with parameter $\lambda$, then

$$\mathbb{E}[X] = \lambda$$

$i=0 \to 0$ $\quad \frac{1}{(i-1)!}$

**Proof.**
$$\mathbb{E}[X] = \sum_{i=0}^{\infty} P(X = i) \cdot i = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!}$$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!}$$

$j = i - 1$

$$= \lambda \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = \lambda \cdot 1 = \lambda$$

= 1 (see prior slides!)

## Variance

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

**Theorem.** If $X$ is a Poisson RV with parameter $\lambda$, then $\mathrm{Var}(X) = \lambda$

**Proof.**

$$\mathbb{E}[X^2] = \sum_{i=0}^{\infty} P(X = i) \cdot i^2 = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} i$$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \cdot i = \lambda \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot (j+1)$$

$$= \lambda \left[ \underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot j}_{= \mathbb{E}[X] = \lambda} + \underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!}}_{= 1} \right] = \lambda^2 + \lambda$$

Similar to the previous proof
Verify offline.

$$\Rightarrow \quad \mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

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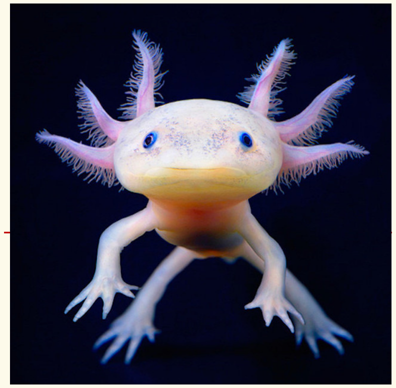
# Poisson Random Variables

**Definition.** A **Poisson random variable** $X$ with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$



Poisson approximates binomial when:

$n$ is very large, $p$ is very small, and $\lambda = np$ is "moderate"

e.g. ($n > 20$ and $p < 0.05$), ($n > 100$ and $p < 0.1$)

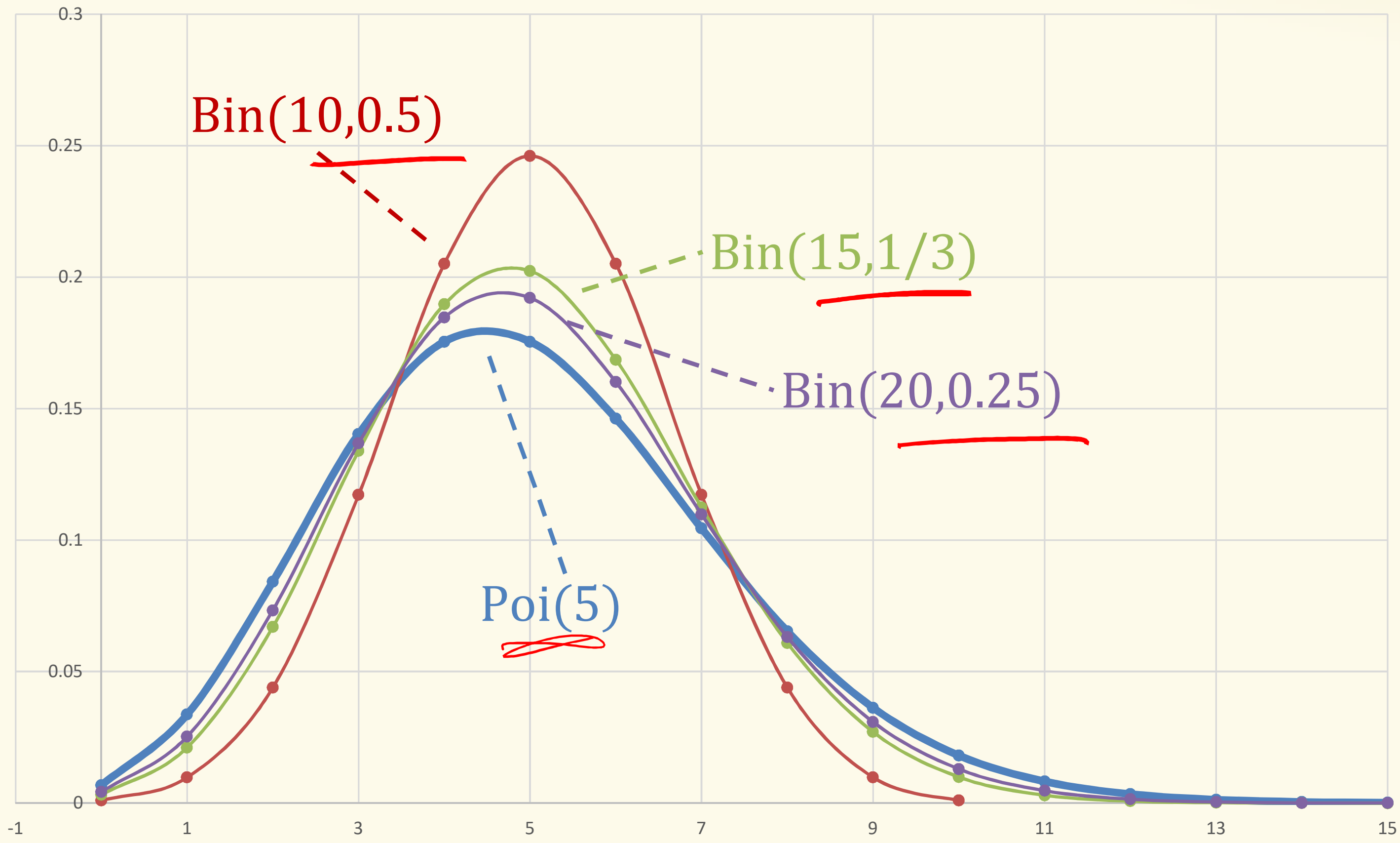
Formally, Binomial approaches Poisson in the limit as $n \to \infty$ (equivalently, $p \to 0$) while holding $np = \lambda$

# Probability Mass Function – Convergence of Binomials

$\lambda = 5$

$p = \frac{5}{n}$

$n = 10{,}15{,}20$




$as\ n \to \infty,\ \ Binomial(n,\ \ p = \ \lambda/n) \to poi(\lambda)$

# From Binomial to Poisson

$X \sim \text{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[X] = np$$

$$\text{Var}(X) = np(1-p)$$

$$n \to \infty$$

$$np = \lambda$$

$$p = \frac{\lambda}{n} \to 0$$

$X \sim \text{Poi}(\lambda)$

$$P(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$E[X] = \lambda$$

$$\text{Var}(X) = \lambda$$

# Example -- Approximate Binomial Using Poisson

Consider sending bit string over a network

- Send bit string of length $n = 10^4$
- Probability of (independent) bit corruption is $p = 10^{-6}$

What is probability that message arrives uncorrupted?

Using $X \sim \text{Poi}(\lambda = np = 10^4 \cdot 10^{-6} = \overbrace{0.01}^{\lambda})$

$$P(X = 0) = e^{-\lambda} \cdot \frac{\lambda^0}{0!} = e^{-0.01} \cdot \frac{0.01^0}{0!} \approx \underline{0.990049834}$$

Using $Y \sim \text{Bin}(\underbrace{10^4}_{}^{n}, \underbrace{10^{-6}}_{}^{p})$

$$P(Y = 0) \approx \underline{0.990049829}$$