CSE 312

Foundations of Computing II

Lecture 12: Zoo of Discrete RVS part II
Poisson Distribution

Announcements

- Midterm info is posted
  - Q&A session next Tuesday 4pm on Zoom
  - Practice midterm + other practice materials posted this Wednesday

Zoo of Random Variables 🦍🐘🦁🐆🐫🦒

$X \sim \text{Unif}(a, b)$

$$P(X = k) = \frac{1}{b - a + 1}$$

$$E[X] = \frac{a + b}{2}$$

$$\text{Var}(X) = \frac{(b - a)(b - a + 2)}{12}$$

$X \sim \text{Ber}(p)$

$$P(X = 1) = p, P(X = 0) = 1 - p$$

$$E[X] = p$$

$$\text{Var}(X) = p(1 - p)$$

$X \sim \text{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$E[X] = np$$

$$\text{Var}(X) = np(1 - p)$$

$X \sim \text{Geo}(p)$

$$P(X = k) = (1 - p)^{k-1} p$$

$$E[X] = \frac{1}{p}$$

$$\text{Var}(X) = \frac{1 - p}{p^2}$$

$X \sim \text{NegBin}(r, p)$

$$P(X = k) = \binom{k - 1}{r - 1} p^r (1 - p)^{k-r}$$

$$E[X] = \frac{r}{p}$$

$$\text{Var}(X) = \frac{r(1 - p)}{p^2}$$

$X \sim \text{HypGeo}(N, K, n)$

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

$$E[X] = n\frac{K}{N}$$

$$\text{Var}(X) = n\frac{K(N - K)(N - n)}{N^2(N - 1)}$$

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Agenda

- **Zoo of Discrete RVs**
  - Uniform Random Variables, Part I
  - Bernoulli Random Variables, Part I
  - Binomial Random Variables, Part I
  - Geometric Random Variables ◀
  - Negative Binomial Random Variables
  - Hypergeometric Random Variables
  - Poisson Distribution
    - Approximate Binomial distribution using Poisson distribution
  - Applications

Geometric Random Variables

A discrete random variable X that models the number of independent trials $Y_i \sim \mathrm{Ber}(p)$ before seeing the first success.

X is called a Geometric random variable with parameter p.

Notation: $X \sim \mathrm{Geo}(p)$

PMF:

Expectation:

Variance:

Examples:
- # of coin flips until first head
- # of random guesses on MC questions until you get one right
- # of random guesses at a password until you hit it

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PMF: $P(X = k) = (1 - p)^{k-1} p$

Expectation: $\mathbb{E}[X] = \dfrac{1}{p}$

Variance: $\text{Var}(X) = \dfrac{1-p}{p^2}$

Examples:
- # of coin flips until first head
- # of random guesses on MC questions until you get one right
- # of random guesses at a password until you hit it

Example: Music Lessons

Your music teacher requires you to play a 1000 note song without mistake. You have been practicing, so you have a probability of 0.999 of getting each note correct (independent of the others). If you mess up a single note in the song, you must start over and play from the beginning. Let X be the number of times you have to play the song from the start. What is $\mathbb{E}[X]$?

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Negative Binomial Random Variables

A discrete random variable X that models the number of independent trials $Y_i \sim \text{Ber}(p)$ before seeing the rth success.
Equivalently, $X = \sum_{i=1}^{r} Z_i$ where $Z_i \sim \text{Geo}(p)$.
X is called a Negative Binomial random variable with parameters r, p.

Notation: $X \sim \text{NegBin}(r, p)$

PMF: $P(X = k) = \binom{k-1}{r-1} p^r (1 - p)^{k-r}$

Expectation: $\mathbb{E}[X] = \dfrac{r}{p}$

Variance: $\text{Var}(X) = \dfrac{r(1-p)}{p^2}$

Hypergeometric Random Variables

A discrete random variable X that models the number of successes in n draws (without replacement) from N items that contain K successes in total. X is called a Hypergeometric RV with parameters N, K, n.

Notation: $X \sim \text{HypGeo}(N, K, n)$

PMF: $P(X = k) = \dfrac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$

Expectation: $\mathbb{E}[X] = n\dfrac{K}{N}$

Variance: $\text{Var}(X) = n\dfrac{K(N-K)(N-n)}{N^2(N-1)}$

Hope you enjoyed the zoo! 🦏🐘🦛🦁🐯🦓🐪🦒

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Preview: Poisson

Model: # events that occur in an hour
- Expect to see 3 events per hour (but will be random)
- The expected number of events in t hours, is $3t$
- Occurrence of events on disjoint time intervals is independent

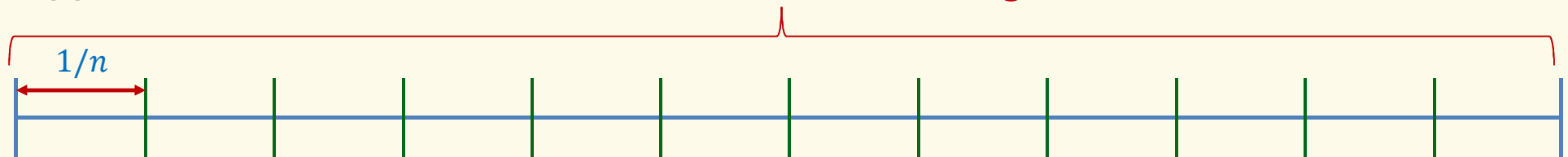**Example – Modelling car arrivals at an intersection**

X = # of cars passing through a light in 1 hour

Example – Model the process of **cars passing through a light in 1 hour**

X = # cars passing through a light in 1 hour. $\mathbb{E}[X] = 3$

Assume: Occurrence of events on disjoint time intervals is independent

Approximation idea: Divide hour into n intervals of length $1/n$

This gives us n independent intervals

Assume at most one car per interval

$p =$ probability car arrives in an interval

What should p be?

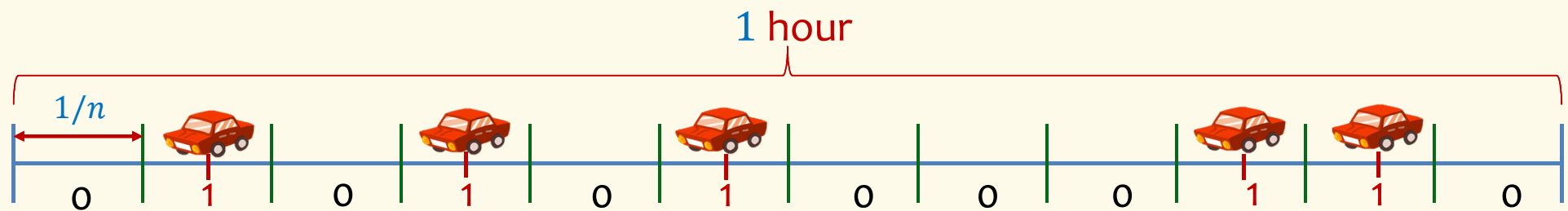**pollev.com/paulbeame028**

A. $3/n$

B. $3n$

C. 3

D. $3/60$

Example – Model the process of **cars passing through a light in 1 hour**

X = # cars passing through a light in 1 hour.   Disjoint time intervals are independent.

Know: $\mathbb{E}[X] = \lambda$ for some given $\lambda > 0$

Discrete version: n intervals, each of length $1/n$.
  In each interval, there is a car with probability $p = \lambda/n$ (assume ≤ 1 car can pass by)
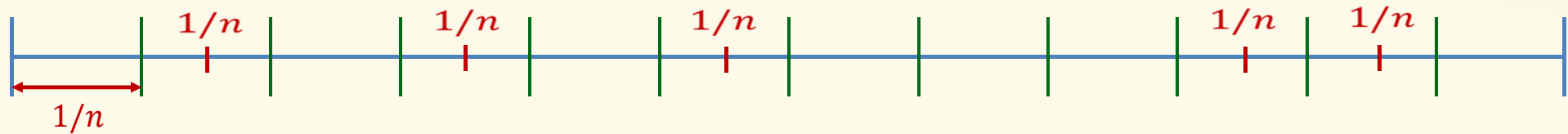
Each interval is Bernoulli: $X_i = 1$ if car in i^{th} interval (0 otherwise). $P(X_i = 1) = \lambda/n$

$$X = \sum_{i=1}^{n} X_i \qquad X \sim \text{Bin}(n, p) \qquad P(X = i) = \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$$

indeed! $\mathbb{E}[X] = pn = \lambda$

Don't like discretization

$1/n$ $1/n$ $1/n$ $1/n$ $1/n$

$1/n$

We want now $n \to \infty$

$$P(X = i) = \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i} = \frac{n!}{(n-i)!\, n^i} \frac{\lambda^i}{i!} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-i}$$

$$\to 1 \qquad\qquad \to e^{-\lambda} \qquad \to 1$$

$$\to P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

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Poisson Distribution

- Suppose "events" happen, independently, at an *average* rate of λ per unit time.

- Let X be the *actual* number of events happening in a given time unit. Then X is a *Poisson* r.v. *with parameter* λ (denoted $X \sim \text{Poi}(\lambda)$) and has distribution (PMF):
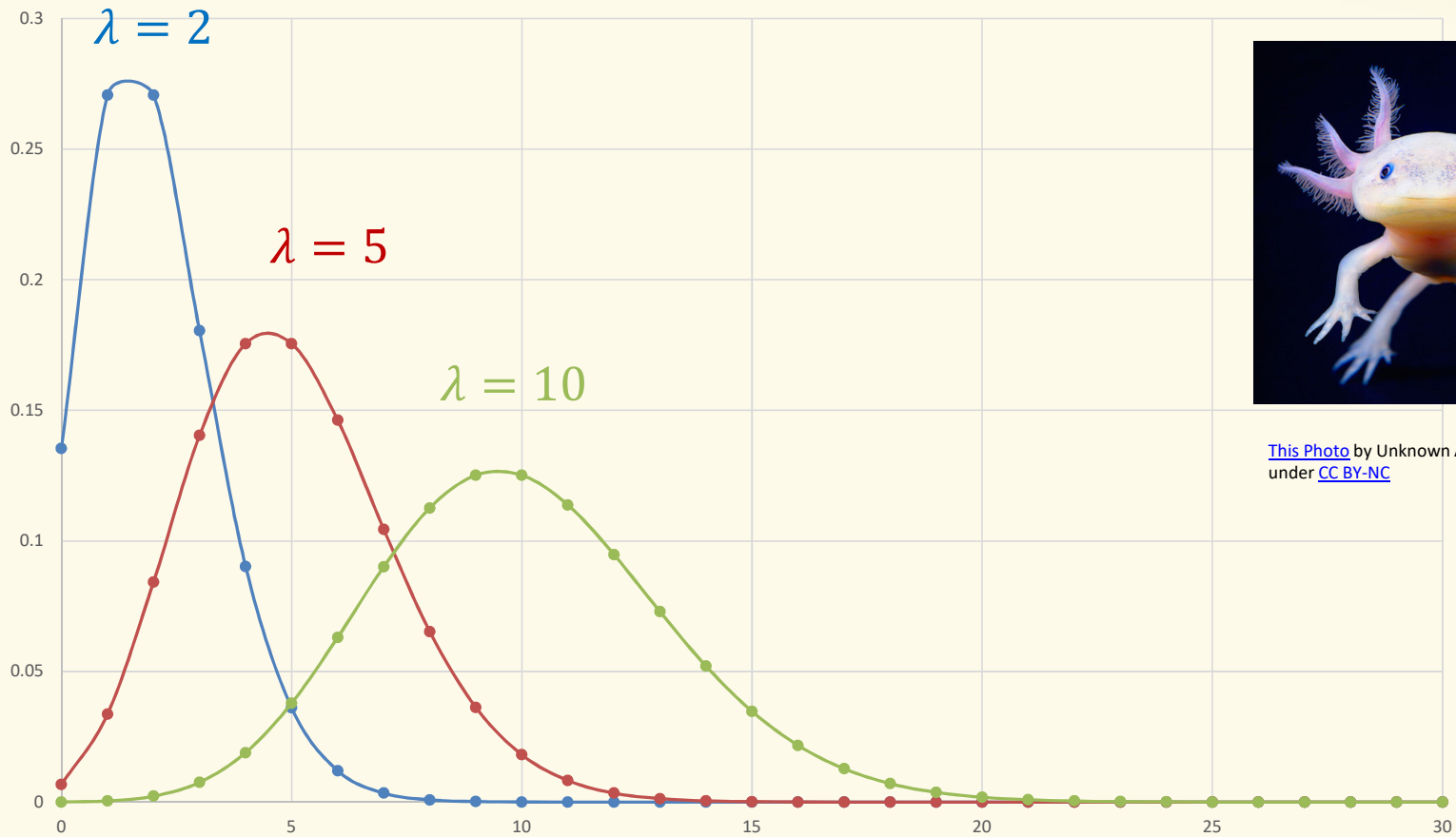
$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Several examples of "Poisson processes":
- # of cars passing through a traffic light <u>in 1 hour</u>
- # of requests to web servers <u>in an hour</u>
- # of photons hitting a light detector <u>in a given interval</u>
- # of patients arriving to ER <u>within an hour</u>

Assume fixed average rate

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Probability Mass Function

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

$\lambda = 2$

$\lambda = 5$

$\lambda = 10$

Validity of Distribution

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

We first want to verify that Poisson probabilities sum up to 1.

$$\sum_{i=0}^{\infty} P(X = i) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1$$

Fact (Taylor series expansion):

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

Expectation

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ, then
$$\mathbb{E}[X] = \lambda$$

Proof.
$$\mathbb{E}[X] = \sum_{i=0}^{\infty} P(X = i) \cdot i = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!}$$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!}$$

$$= \lambda \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = \lambda \cdot 1 = \lambda$$

= 1 (see prior slides!)

Variance

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ, then $\mathrm{Var}(X) = \lambda$

Proof. $\mathbb{E}[X^2] = \sum_{i=0}^{\infty} P(X = i) \cdot i^2 = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} i$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \cdot i = \lambda \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot (j+1)$$

$$= \lambda \left[\underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot j}_{= \mathbb{E}[X] = \lambda} + \underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!}}_{= 1} \right] = \lambda^2 + \lambda$$

Similar to the previous proof
Verify offline.

$$\Rightarrow \quad \mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

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Poisson Random Variables

Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

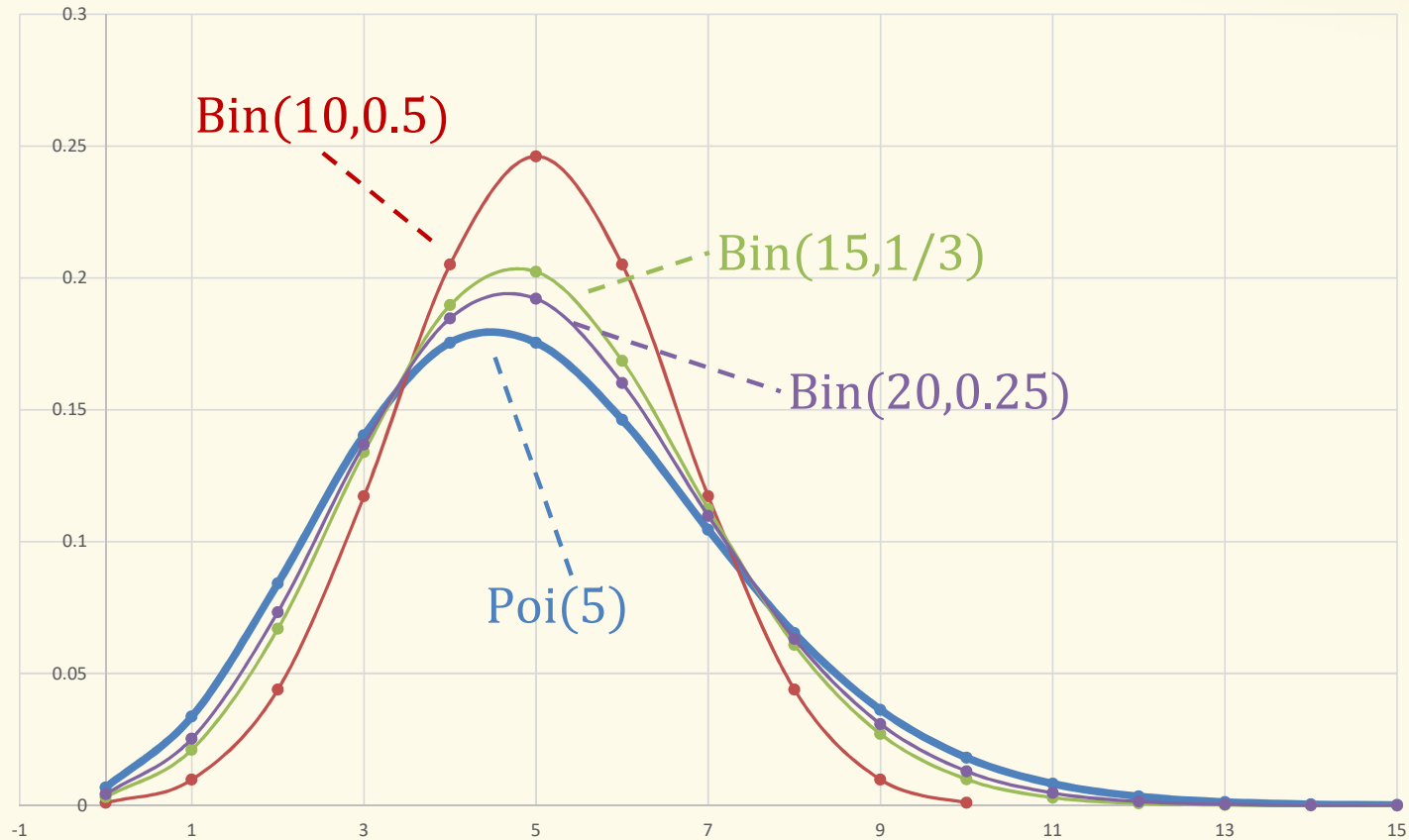$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Poisson approximates binomial when:
n is very large, p is very small, and $\lambda = np$ is "moderate"
e.g. $(n > 20$ and $p < 0.05$), ($n > 100$ and $p < 0.1$)

Formally, Binomial approaches Poisson in the limit as
$n \rightarrow \infty$ (equivalently, $p \rightarrow 0$) while holding $np = \lambda$

Probability Mass Function – Convergence of Binomials

$\lambda = 5$

$p = \dfrac{5}{n}$

$n = 10,15,20$

Bin(10,0.5)

Bin(15,1/3)

Bin(20,0.25)

Poi(5)

$as\ n \to \infty,\ Binomial(n,\ p = \lambda/n) \to poi(\lambda)$

From Binomial to Poisson

$$X \sim \text{Bin}(n, p)$$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[X] = np$$

$$\text{Var}(X) = np(1-p)$$

$$n \to \infty$$
$$np = \lambda$$
$$p = \frac{\lambda}{n} \to 0$$

$$X \sim \text{Poi}(\lambda)$$

$$P(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$E[X] = \lambda$$

$$\text{Var}(X) = \lambda$$

Example -- Approximate Binomial Using Poisson

Consider sending bit string over a network
- Send bit string of length $n = 10^4$
- Probability of (independent) bit corruption is $p = 10^{-6}$

What is probability that message arrives uncorrupted?

Using $X \sim \text{Poi}(\lambda = np = 10^4 \cdot 10^{-6} = 0.01)$

$$P(X = 0) = e^{-\lambda} \cdot \frac{\lambda^0}{0!} = e^{-0.01} \cdot \frac{0.01^0}{0!} \approx 0.990049834$$

Using $Y \sim \text{Bin}(10^4, 10^{-6})$

$$P(Y = 0) \approx 0.990049829$$

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Sum of Independent Poisson RVs

Theorem. Let $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.
Let $Z = X + Y$. For all $z = 0,1,2,3 \dots,$

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

More generally, let $X_1 \sim \text{Poi}(\lambda_1), \cdots, X_n \sim \text{Poi}(\lambda_n)$ such that $\lambda = \Sigma_i \lambda_i$.
Let $Z = \Sigma_i X_i$

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

Sum of Independent Poisson RVs

Theorem. Let $X \sim \mathrm{Poi}(\lambda_1)$ and $Y \sim \mathrm{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.
Let $Z = X + Y$. For all $z = 0,1,2,3 \ldots$,

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

$P(Z = z) = ?$

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1. $P(Z = z) = \Sigma_{j=0}^{z} P(X = j, Y = z - j)$
2. $P(Z = z) = \Sigma_{j=0}^{\infty} P(X = j, Y = z - j)$
3. $P(Z = z) = \Sigma_{j=0}^{z} P(Y = z - j | X = j)\, P(X = j)$
4. $P(Z = z) = \Sigma_{j=0}^{z} P(Y = z - j | X = j)$

A. All of them are right
B. The first 3 are right
C. Only 1 is right
D. Don't know

Proof

$$P(Z = z) = \Sigma_{j=0}^{z} P(X = j, Y = z - j)$$ Law of total probability

$$= \Sigma_{j=0}^{z} P(X = j) \, P(Y = z - j) = \Sigma_{j=0}^{z} \; e^{-\lambda_1} \cdot \frac{\lambda_1^{j}}{j!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{z-j}}{z - j!}$$ Independence

$$= e^{-\lambda_1 - \lambda_2} \left(\Sigma_{j=0}^{z} \; \cdot \frac{1}{j! \, z - j!} \cdot \lambda_1^{j} \lambda_2^{z-j} \right)$$

$$= e^{-\lambda} \left(\Sigma_{j=0}^{z} \frac{z!}{j! \, z - j!} \cdot \lambda_1^{j} \lambda_2^{z-j} \right) \frac{1}{z!}$$

$$= e^{-\lambda} \cdot (\lambda_1 + \lambda_2)^{z} \cdot \frac{1}{z!} = e^{-\lambda} \cdot \lambda^{z} \cdot \frac{1}{z!}$$ Binomial
Theorem

Poisson Random Variables

Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

General principle:

- Events happen at an average rate of λ per time unit
- Number of events happening at a time unit X is distributed according to Poi(λ)

- Poisson approximates Binomial when n is large, p is small, and np is moderate
- Sum of independent Poisson is still a Poisson