CSE 312

Foundations of Computing II

Lecture 15: Normal Distribution & Central Limit Theorem

Midterm Wed in class. See Edstem

OH Today after class.

Midterm Q&A Session tomorrow

4:00 pm

Zoom: link later

Review Continuous RVs

Probability Density Function (PDF).

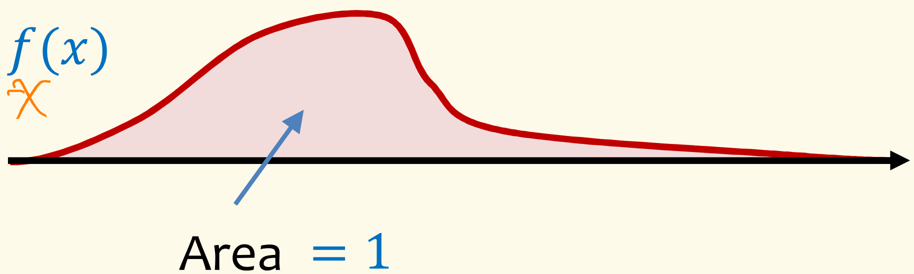
$f: \mathbb{R} \to \mathbb{R}$ s.t.

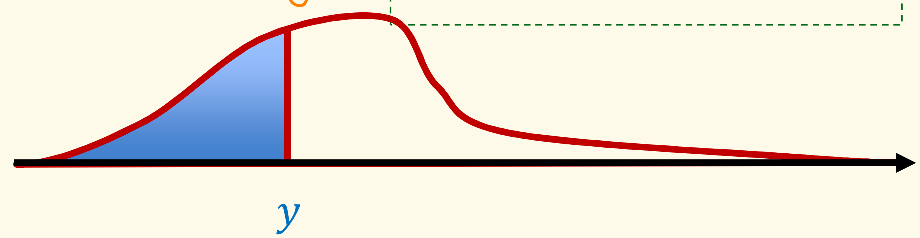
- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 1$

Cumulative Distribution Function (CDF).

$$F(y) = \int_{-\infty}^{y} f(x)\,\mathrm{d}x$$

Theorem. $f(x) = \frac{dF(x)}{dx}$

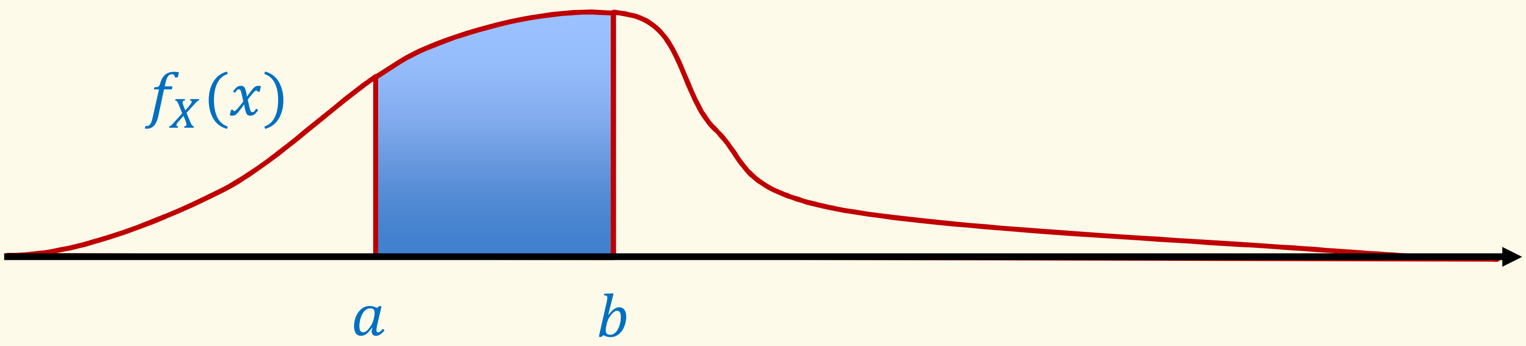




Density ≠ Probability !

$F_X(y) = P(X \leq y)$

Review Continuous RVs



$$P(X \in [a, b]) = \int_a^b f_X(x)\mathrm{d}x = F_X(b) - F_X(a)$$

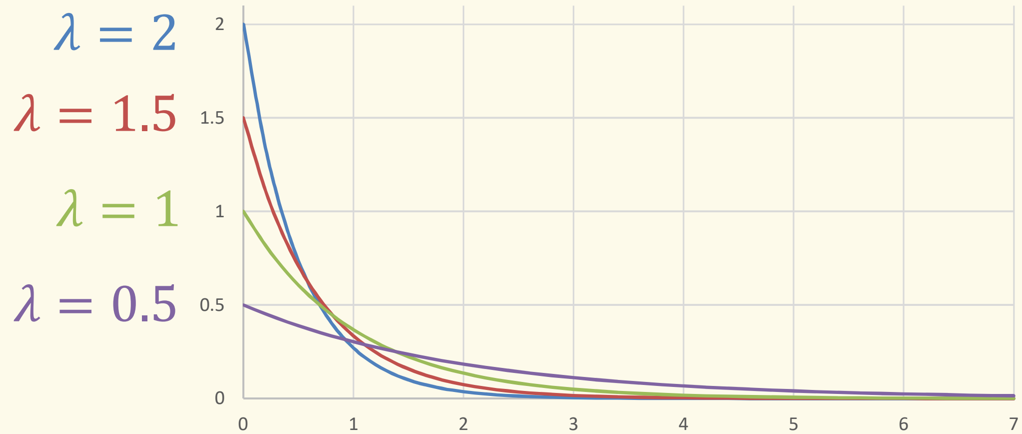
Review Exponential Distribution

Definition. An **exponential random variable** X with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write $X \sim \mathrm{Exp}(\lambda)$ and say X that follows the exponential distribution.

CDF: For $y \geq 0$,
$F_X(y) = 1 - e^{-\lambda y}$



Agenda

- Normal Distribution
- Practice with Normals
- Central Limit Theorem (CLT)

The Normal Distribution

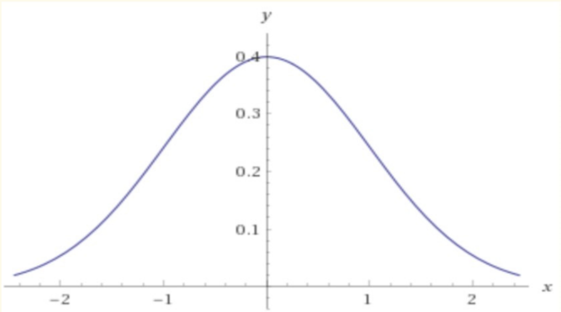
Definition. A **Gaussian (or normal) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Carl Friedrich Gauss

$\sqrt{2\pi} = \int_{-\infty}^{\infty} e^{-x^2/2} dx$

We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$.



$\mu \quad \sigma^2$

$\mathcal{N}(0, 1)$.

The Normal Distribution

Carl Friedrich Gauss

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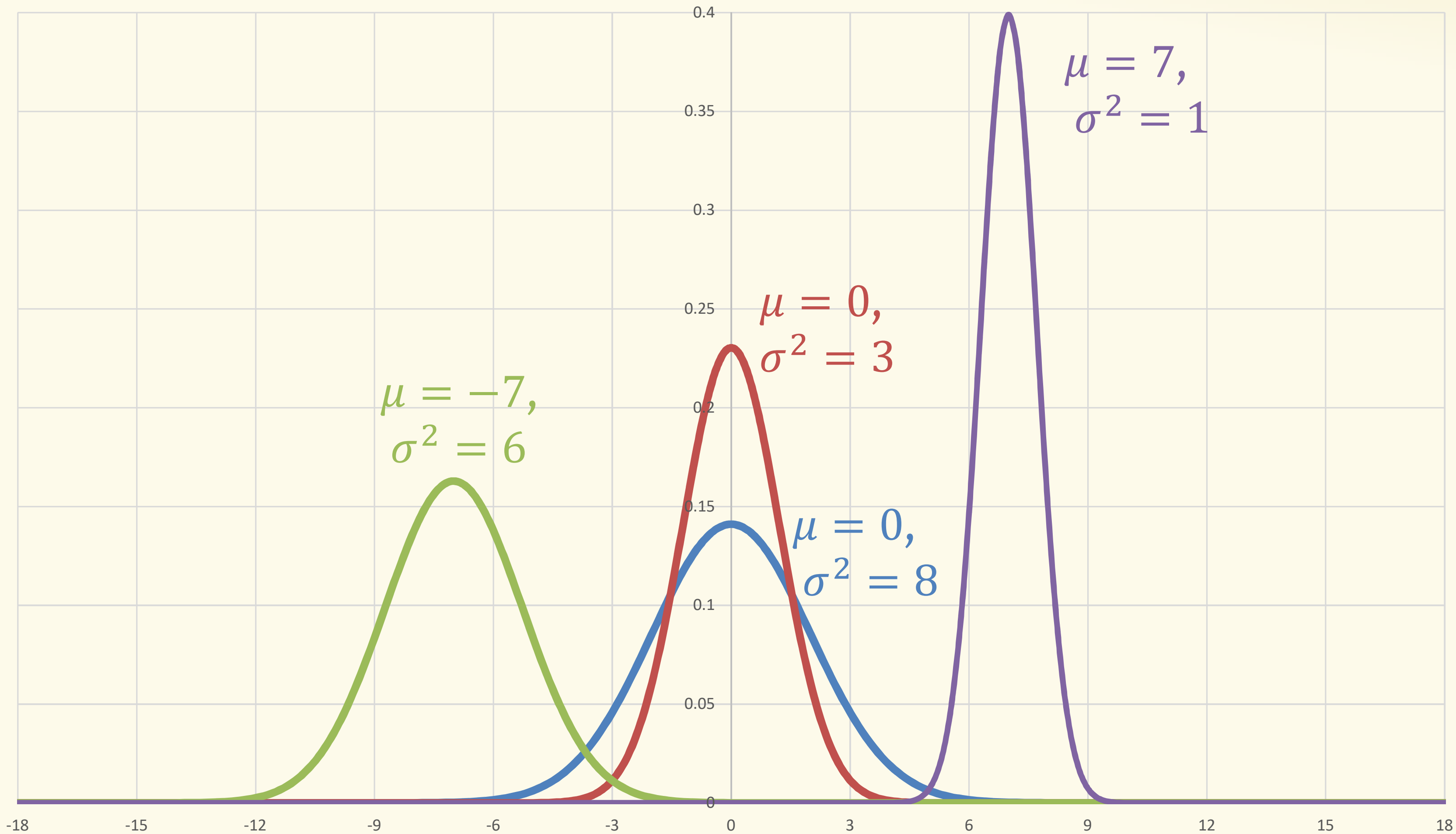
Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}[X] = \mu$, and $\mathrm{Var}(X) = \sigma^2$

Proof of expectation is easy because density curve is symmetric around μ, $f_X(\mu - x) = f_X(\mu + x)$, but proof for variance requires integration of $e^{-x^2/2}$

The Normal Distribution

Aka a “Bell Curve” (imprecise name)



Closure of normal distribution – Under Shifting and Scaling

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Proof. $\mathbb{E}[Y] = a\,\mathbb{E}[X] + b = a\mu + b$

$\mathrm{Var}(Y) = a^2\,\mathrm{Var}(X) = a^2\sigma^2$

Can show with algebra that the PDF of $Y = aX + b$ is still normal.

Note: $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$

$a = \frac{1}{\sigma}$

$\mathbb{E}\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}\mathbb{E}[X-\mu] = \frac{1}{\sigma}(\underbrace{\mathbb{E}[X]}_{\mu}-\mu) = 0$

CDF of normal distribution

$\mathcal{N}(0,1)$

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Standard (unit) normal = $\mathcal{N}(0, 1)$

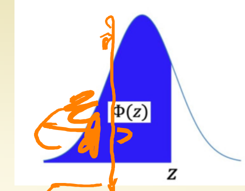
CDF. $\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \mathrm{d}x$ for $Z \sim \mathcal{N}(0, 1)$

Note: $\Phi(z)$ has no closed form – generally given via tables

Z

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F_X(z) = P(X \leq z) = P\left(\frac{X-\mu}{\sigma} \leq \frac{z-\mu}{\sigma}\right) = \Phi(\frac{z-\mu}{\sigma})$

Table of Standard Cumulative Normal Density



$P(Z \leq 1.09) = \Phi(1.09) \approx 0.8621$

What is $P(Z \leq -1.09)$? $= \Phi(-1.09)$ $= 1 - \Phi(1.09)$

Poll: pollev.com/paulbeame028

a. 0.1379
b. 0.8621
c. 0
d. Not able to compute

Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Closure of the normal -- under addition

Fact. If $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ (both independent normal RV)
then $aX + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$

μ

Note: The special thing is that **the sum of normal RVs is still a normal RV.**
The values of the expectation and variance are **not** surprising.

Why not surprising?

- Linearity of expectation (always true)
- When X and Y are independent, $\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$

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What about Non-standard normal?

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$

Z

Therefore,

$$F_X(z) = P(X \leq z) = P\left(\frac{X-\mu}{\sigma} \leq \frac{z-\mu}{\sigma}\right) = \Phi\left(\frac{z-\mu}{\sigma}\right)$$

Z

Example

$\mu = 0.4$ $\qquad \frac{X-\mu}{\sigma}$

Let $X \sim \mathcal{N}(0.4, 4 = 2^2)$.

$\sigma = 2 \quad \sigma^2$ $\qquad \frac{.8}{2} = .4$

$$P(X \leq 1.2) = P\left(\frac{X - 0.4}{2} \leq \frac{1.2 - 0.4}{2}\right)$$

$$= P\left(\frac{X - 0.4}{2} \leq 0.4\right) = \Phi(0.4) \approx 0.6554$$

$\sim \mathcal{N}(0,1)$

0.1	0.5398	0.5438
0.2	0.5793	0.5832
0.3	0.6179	0.6217
0.4	0.6554	0.6591
0.5	0.6915	0.6950
0.6	0.7257	0.7291
0.7	0.7580	0.7611

Example

$\mu = 3$
$\sigma = 4$

Let $X \sim \mathcal{N}(3, 16)$.

$$P(2 < X < 5) = P\left(\frac{2\ -3}{4} < \frac{X-3}{4} < \frac{5\ -3}{4}\right)$$

$$= P\left(-\frac{1}{4} < Z < \frac{1}{2}\right)$$

$Z \sim N(0,1)$

$$= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{4}\right)$$

$1 - \Phi\left(\frac{1}{4}\right)$
$\| $
$\Phi\left(\frac{1}{2}\right) - \overline{\Phi\left(-\frac{1}{4}\right)}$

$$= \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{4}\right)\right) \approx 0.29017$$

Example – How Many Standard Deviations Away?

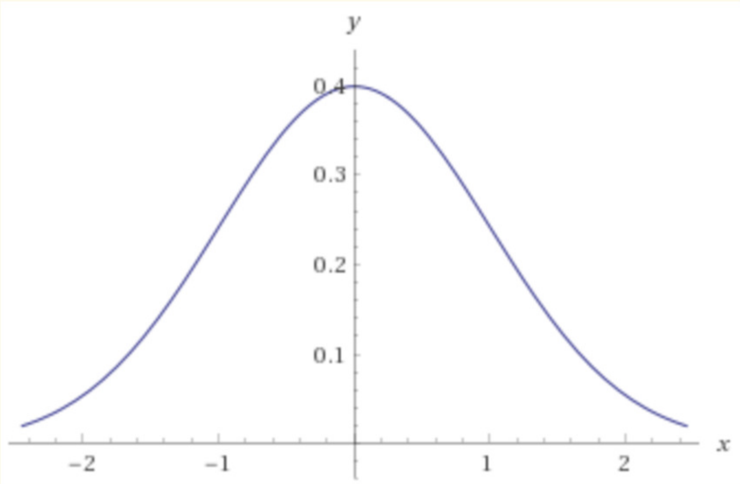
Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

$$P(|X-\mu| < k\sigma) = P\left(\frac{|X-\mu|}{\sigma} < k\right) =$$

$$= P\left(-k < \frac{X-\mu}{\sigma} < k\right) = \Phi(k) - \Phi(-k)$$

e.g. $k \doteq 1$: 68%
$k \doteq 2$: 95%
$k \doteq 3$: 99%

Halloween Brain Break



Normal Distribution

Paranormal Distribution

Agenda

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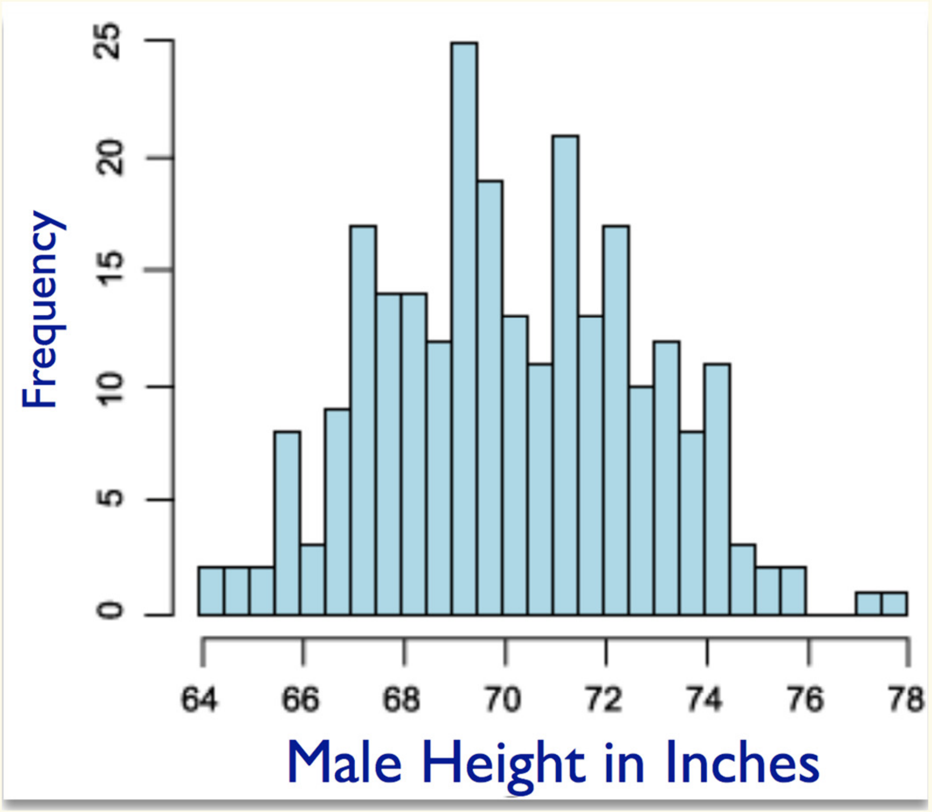
Gaussian in Nature

Empirical distribution of collected data often resembles a Gaussian …



e.g. Height distribution resembles Gaussian.

R.A.Fisher (1918) observed that the height is likely the outcome of the sum of many independent random parameters, i.e., can written as

$$X = X_1 + \cdots + X_n$$

Sum of Independent RVs

i.i.d. = independent and identically distributed

$X_1, \dots, X_n$ i.i.d. with expectation μ and variance σ^2

Define

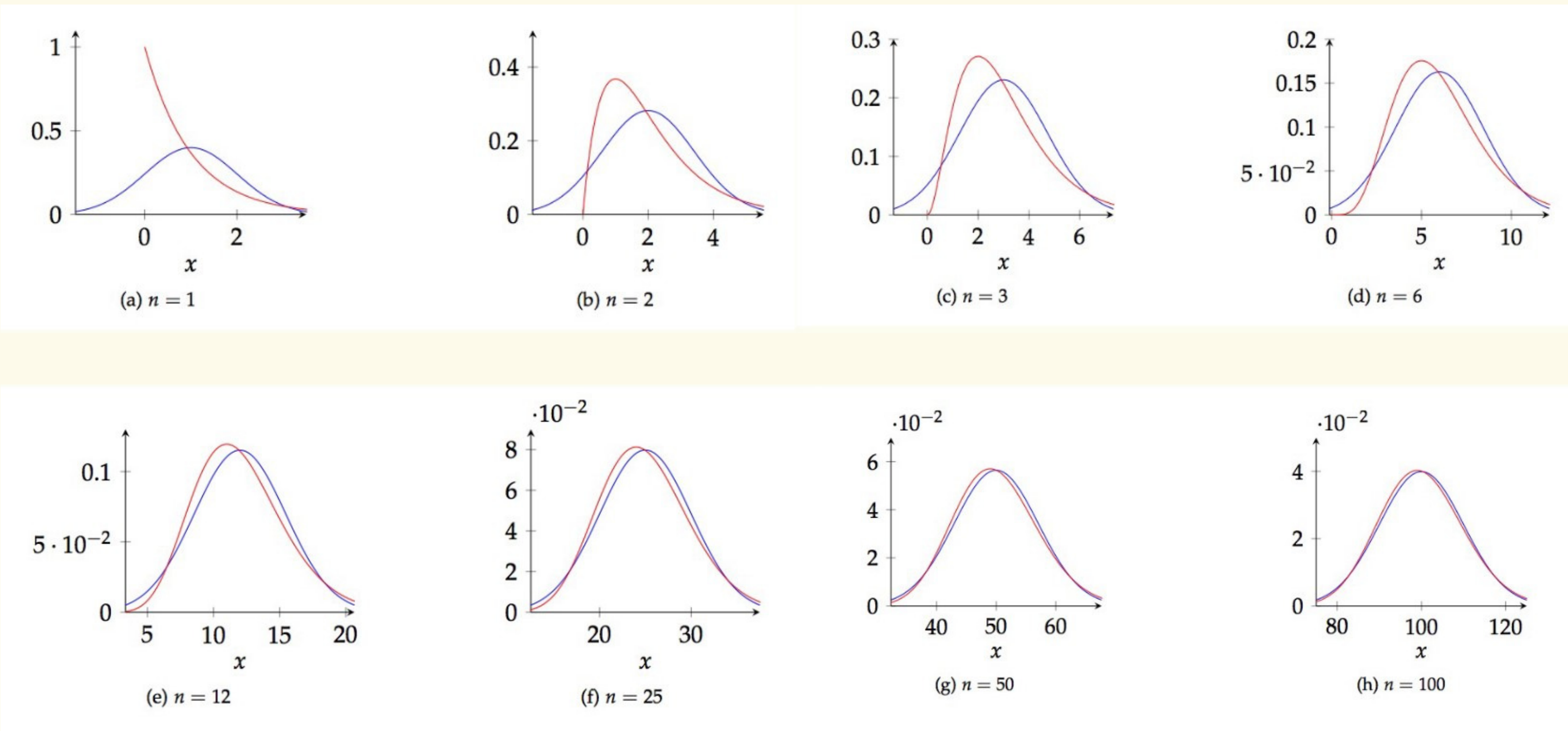
$$S_n = X_1 + \cdots + X_n$$

$$\mathbb{E}[S_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = n\mu$$

$$\mathrm{Var}(S_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) = n\sigma^2$$

Empirical observation: S_n looks like a normal RV as n grows.

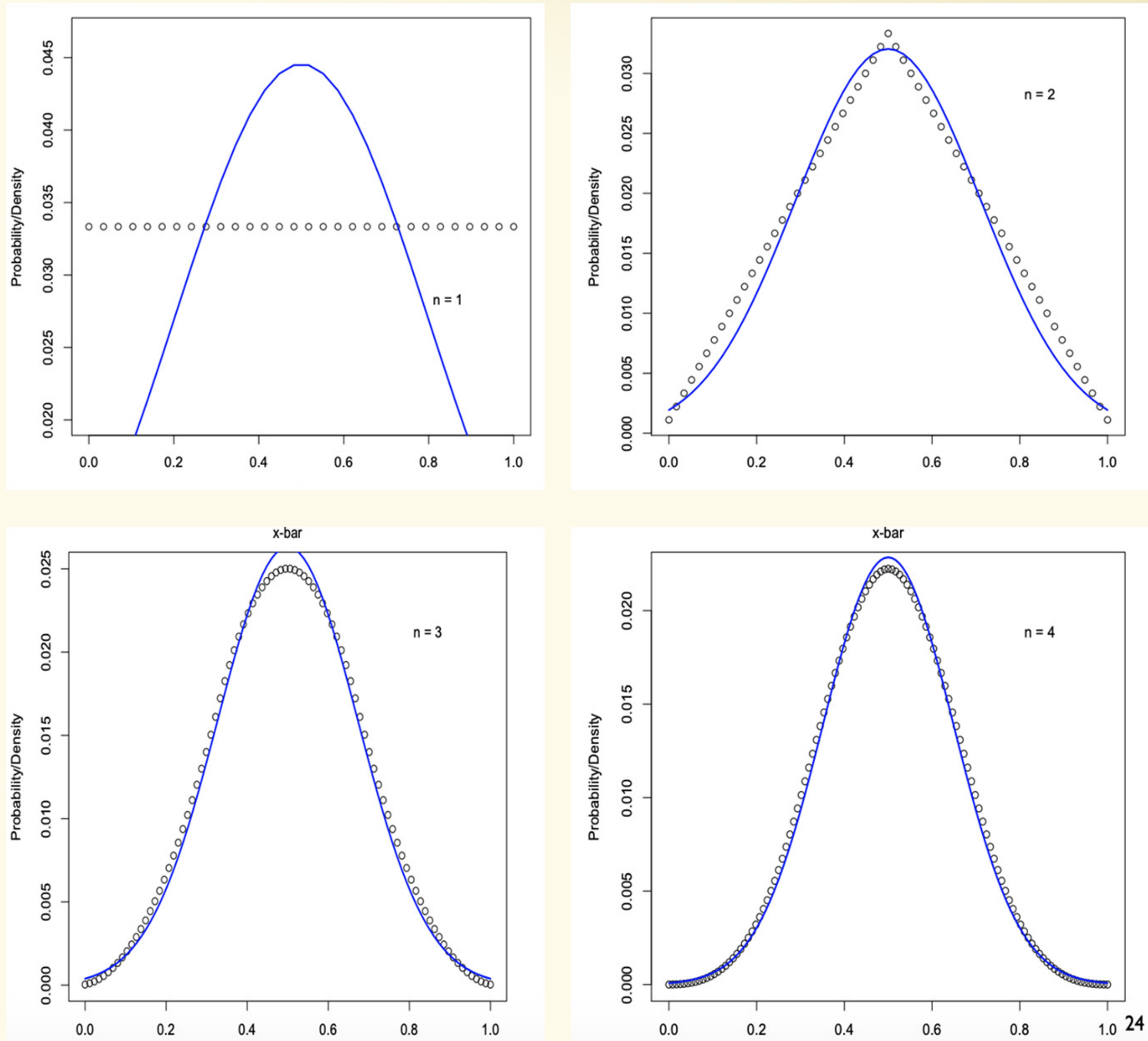
Example: Sum of n i.i.d. $\mathrm{Exp}(1)$ random variables



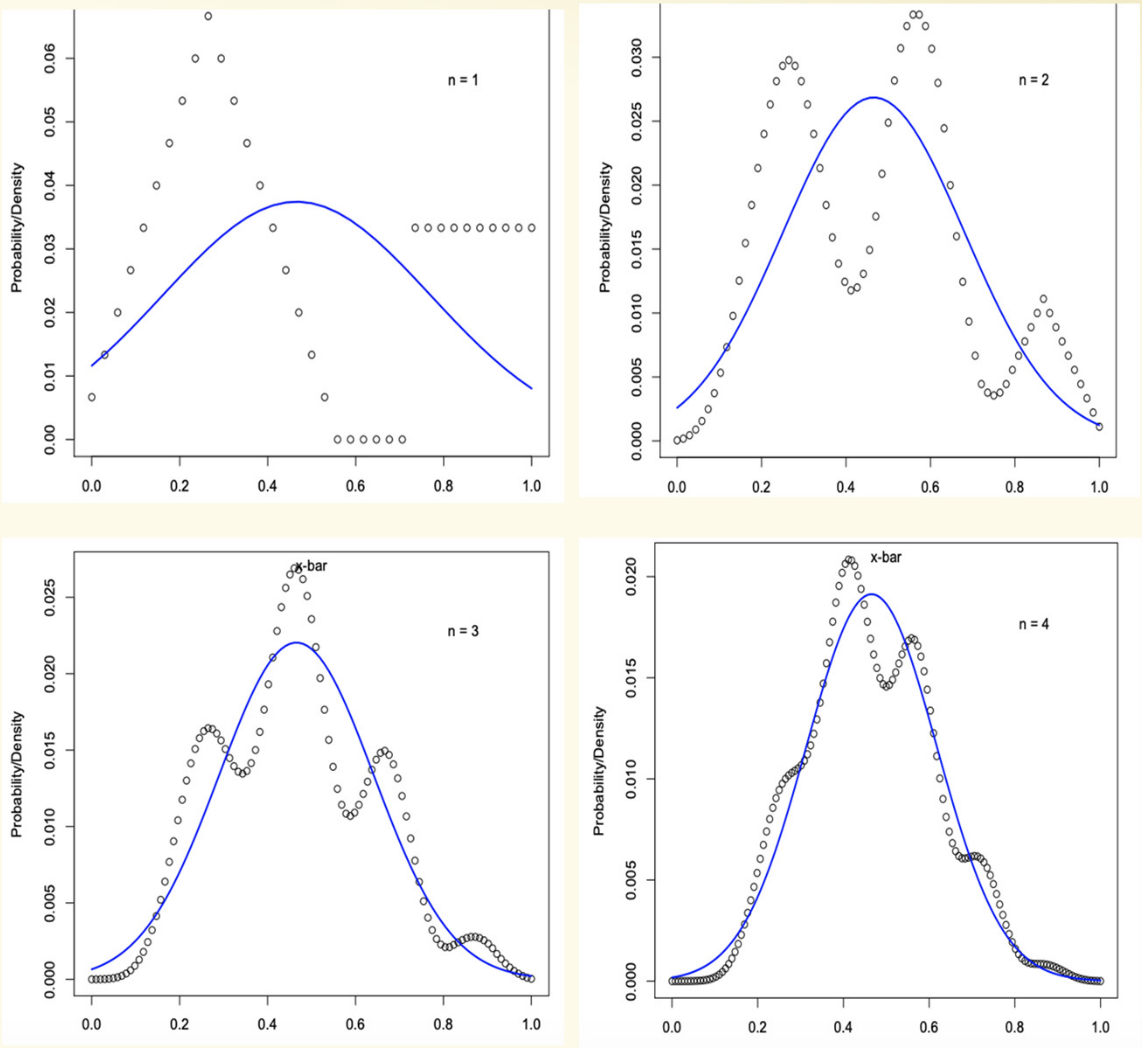
(a) $n = 1$ (b) $n = 2$ (c) $n = 3$ (d) $n = 6$

(e) $n = 12$ (f) $n = 25$ (g) $n = 50$ (h) $n = 100$

CLT (Idea)



CLT (Idea)



n = 1
n = 2
n = 3
n = 4
x-bar
x-bar
Probability/Density
Probability/Density
Probability/Density
Probability/Density

Central Limit Theorem

$X_1, \dots, X_n$ i.i.d., each with expectation μ and variance σ^2

$\sigma_{new}^2 = n\sigma^2$

Define $S_n = X_1 + \cdots + X_n$ and

$\sigma_{new} = \sqrt{n} \cdot \sigma$

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$$\mathbb{E}[Y_n] = \frac{1}{\sigma\sqrt{n}}(\mathbb{E}[S_n] - n\mu) = \frac{1}{\sigma\sqrt{n}}(n\mu - n\mu) = 0$$

$$\mathrm{Var}(Y_n) = \frac{1}{\sigma^2 n}\big(\mathrm{Var}(S_n - n\mu)\big) = \frac{\mathrm{Var}(S_n)}{\sigma^2 n} = \frac{\sigma^2 n}{\sigma^2 n} = 1$$

Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

Theorem. (Central Limit Theorem) The CDF of Y_n converges to the CDF of the standard normal $\mathcal{N}(0,1)$, i.e.,

$$\underline{\lim_{n\to\infty} P(Y_n \le y)} = \underline{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-x^2/2}\mathrm{d}x} \quad \Phi(y)$$

Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

Theorem. (Central Limit Theorem) The CDF of Y_n converges to the CDF of the standard normal $\mathcal{N}(0,1)$, i.e.,

$$\lim_{n\to\infty} P(Y_n \leq y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \mathrm{d}x$$

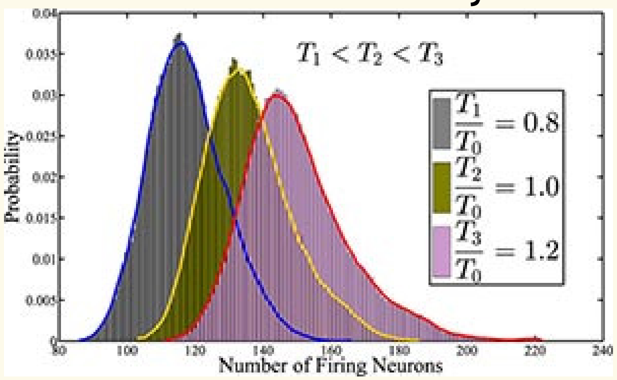
$\frac{n\sigma^2}{n^2}$

Also stated as:

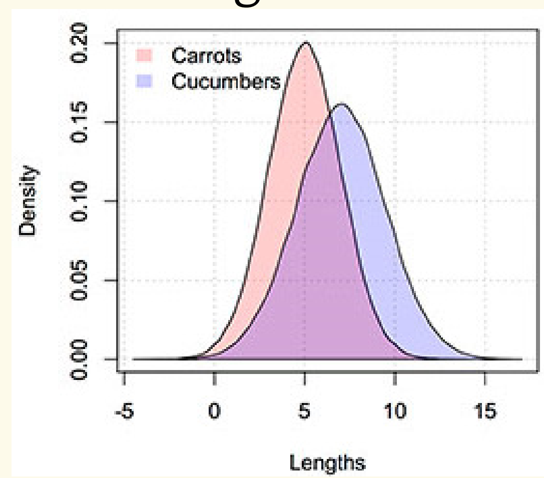
- $\lim_{n\to\infty} Y_n \to \mathcal{N}(0,1)$ $\leftarrow$
- $\lim_{n\to\infty} \underline{\frac{1}{n}\sum_{i=1}^{n} X_i} \to \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$ for $\mu = \mathbb{E}[X_i]$ and $\sigma^2 = \mathrm{Var}(X_i)$

CLT → Normal Distribution EVERYWHERE

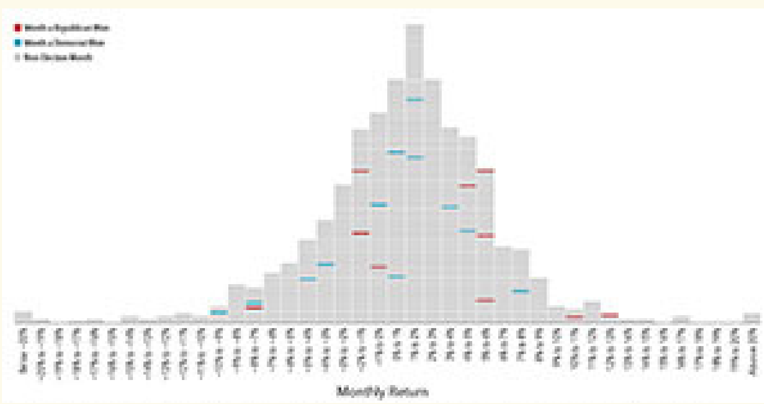
Neuron Activity



Vegetables



S&P 500 Returns after Elections



Examples from:
https://galtonboard.com/probabilityexamplesinlife