

CSE 312

Foundations of Computing II

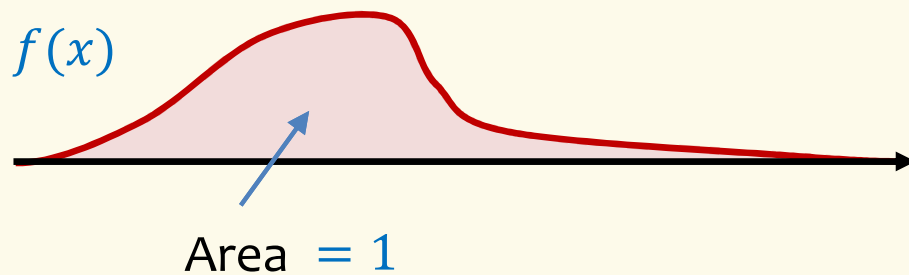
Lecture 15: Normal Distribution & Central Limit Theorem

Review Continuous RVs

Probability Density Function (PDF).

$f: \mathbb{R} \rightarrow \mathbb{R}$ s.t.

- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x) dx = 1$

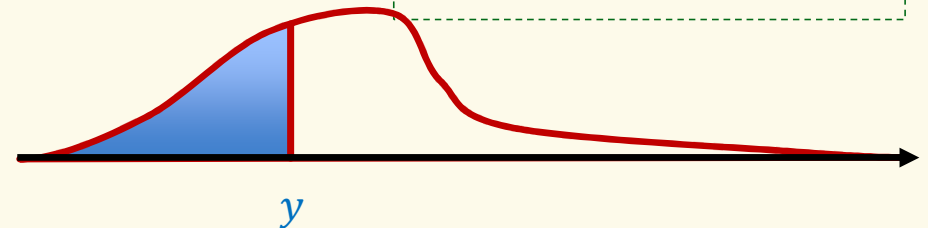


Density \neq Probability !

Cumulative Distribution Function (CDF).

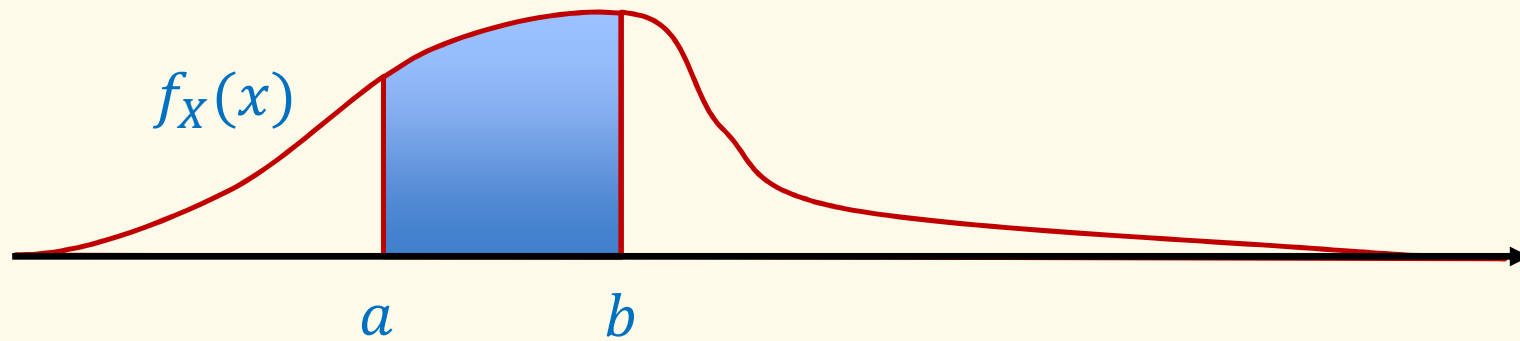
$$F(y) = \int_{-\infty}^y f(x) dx$$

Theorem. $f(x) = \frac{dF(x)}{dx}$



$$F_X(y) = P(X \leq y)$$

Review Continuous RVs



$$P(X \in [a, b]) = \int_a^b f_X(x) dx = F_X(b) - F_X(a)$$

Review Exponential Distribution

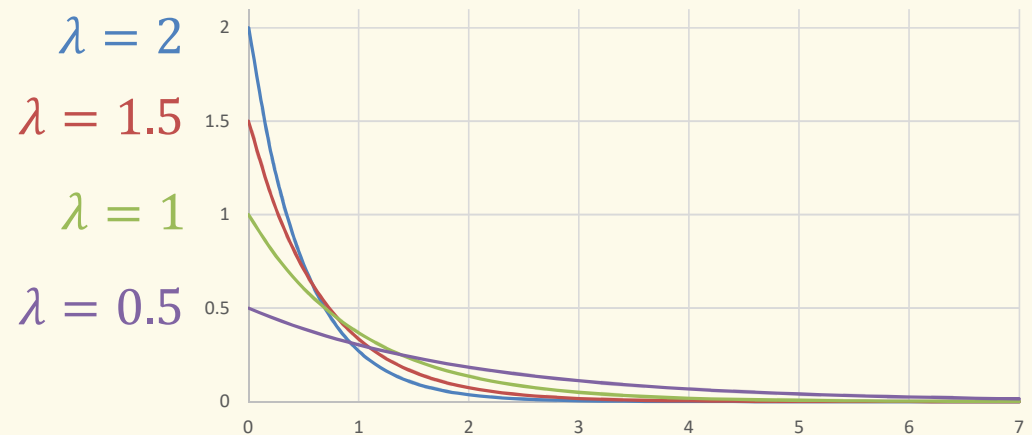
Definition. An **exponential random variable** X with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write $X \sim \text{Exp}(\lambda)$ and say X that follows the exponential distribution.

CDF: For $y \geq 0$,

$$F_X(y) = 1 - e^{-\lambda y}$$



Agenda

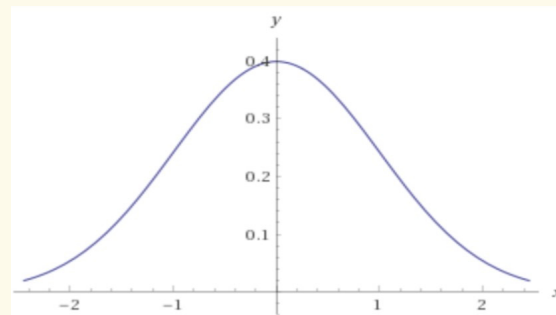
- Normal Distribution ◀
- Practice with Normals
- Central Limit Theorem (CLT)

The Normal Distribution

Definition. A **Gaussian (or normal)** random variable with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$.



$\mathcal{N}(0, 1)$.



Carl Friedrich
Gauss

The Normal Distribution

Definition. A **Gaussian (or normal)** random variable with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$.

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}[X] = \mu$, and $\text{Var}(X) = \sigma^2$

Proof of expectation is easy because density curve is symmetric around μ ,

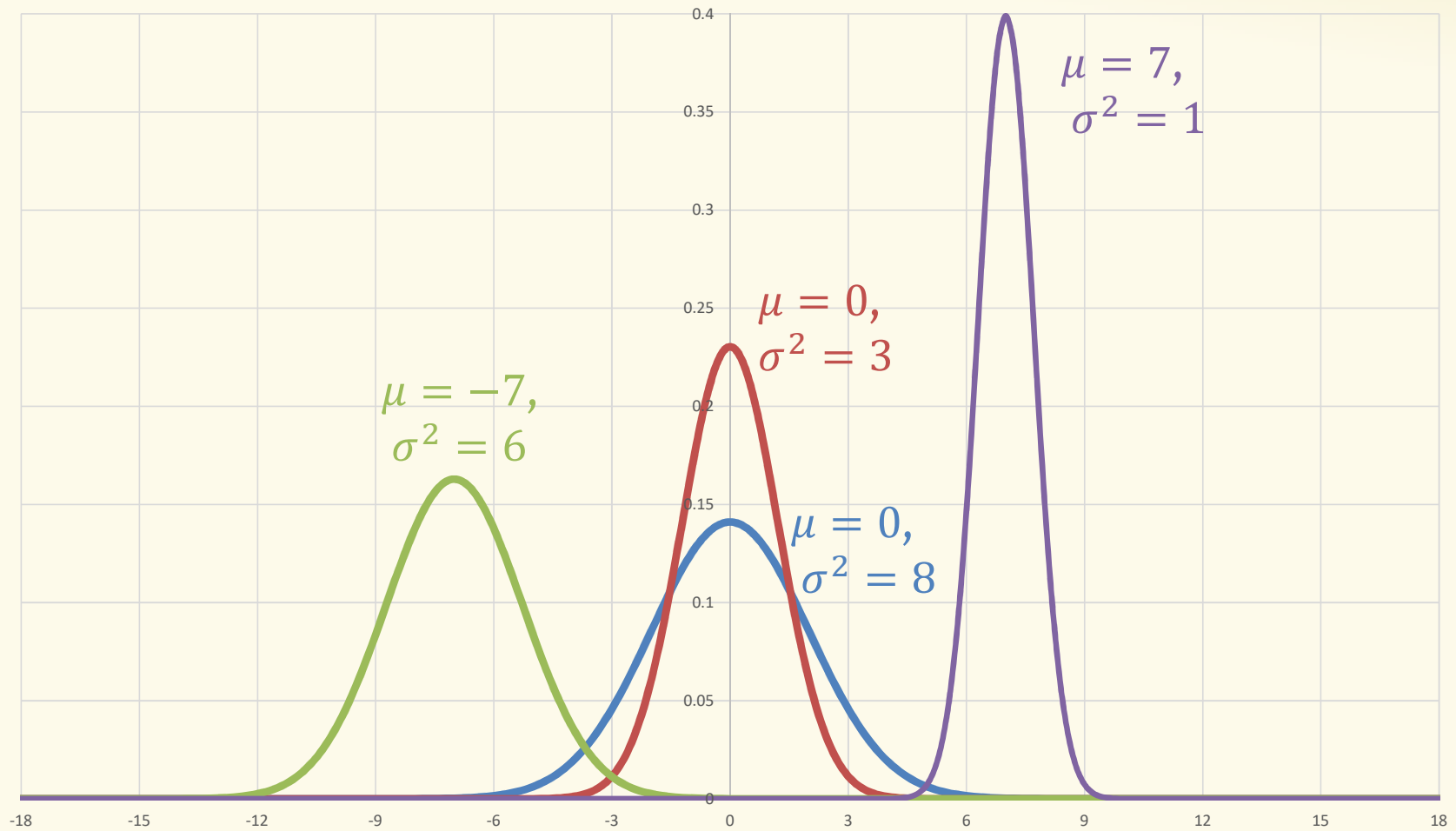
$f_X(\mu - x) = f_X(\mu + x)$, but proof for variance requires integration of $e^{-x^2/2}$



Carl Friedrich
Gauss

The Normal Distribution

Aka a “Bell Curve” (imprecise name)



Closure of normal distribution – Under Shifting and Scaling

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Proof. $\mathbb{E}[Y] = a \mathbb{E}[X] + b = a\mu + b$

$$\text{Var}(Y) = a^2 \text{Var}(X) = a^2\sigma^2$$

Can show with algebra that the PDF of $Y = aX + b$ is still normal.

Note: $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$

CDF of normal distribution

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

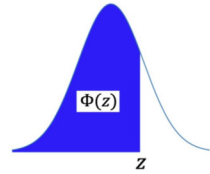
Standard (unit) normal = $\mathcal{N}(0, 1)$

CDF. $\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-x^2/2} dx$ for $Z \sim \mathcal{N}(0, 1)$

Note: $\Phi(z)$ has no closed form – generally given via tables

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F_X(z) = P(X \leq z) = P\left(\frac{X-\mu}{\sigma} \leq \frac{z-\mu}{\sigma}\right) = \Phi\left(\frac{z-\mu}{\sigma}\right)$

Table of Standard Cumulative Normal Density



$$P(Z \leq 1.09) = \Phi(1.09) \approx 0.8621$$

What is
 $P(Z \leq -1.09)$?

Poll:

pollev.com/paulbeameo28

- a. 0.1379
- b. 0.8621
- c. 0
- d. Not able to compute

Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0, 1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Closure of the normal -- under addition

Fact. If $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ (both independent normal RV) then $aX + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$

Note: The special thing is that **the sum of normal RVs is still a normal RV.**
The values of the expectation and variance are **not** surprising.

Why not surprising?

- Linearity of expectation (always true)
- When X and Y are independent, $\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y)$

Agenda

- Normal Distribution
- Practice with Normals ◀
- Central Limit Theorem (CLT)

What about Non-standard normal?

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$

Therefore,

$$F_X(z) = P(X \leq z) = P\left(\frac{X - \mu}{\sigma} \leq \frac{z - \mu}{\sigma}\right) = \Phi\left(\frac{z - \mu}{\sigma}\right)$$

Example

Let $X \sim \mathcal{N}(0.4, 4 = 2^2)$.

$$\begin{aligned} P(X \leq 1.2) &= P\left(\frac{X - 0.4}{2} \leq \frac{1.2 - 0.4}{2}\right) \\ &= P\left(\frac{X - 0.4}{2} \leq 0.4\right) = \Phi(0.4) \approx 0.6554 \end{aligned}$$

$\sim \mathcal{N}(0, 1)$

0.1	0.5398	0.5438
0.2	0.5793	0.5832
0.3	0.6179	0.6217
0.4	0.6554	0.6591
0.5	0.6915	0.6950
0.6	0.7257	0.7291
0.7	0.7580	0.7611

Example

Let $X \sim \mathcal{N}(3, 16)$.

$$\begin{aligned} P(2 < X < 5) &= P\left(\frac{2 - 3}{4} < \frac{X - 3}{4} < \frac{5 - 3}{4}\right) \\ &= P\left(-\frac{1}{4} < Z < \frac{1}{2}\right) \\ &= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{4}\right) \\ &= \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{4}\right)\right) \approx 0.29017 \end{aligned}$$

Example – How Many Standard Deviations Away?

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

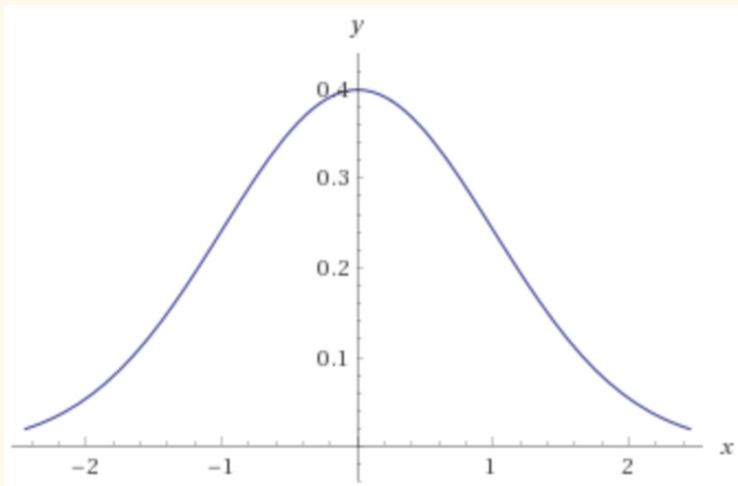
$$\begin{aligned} P(|X - \mu| < k\sigma) &= P\left(\frac{|X - \mu|}{\sigma} < k\right) = \\ &= P\left(-k < \frac{X - \mu}{\sigma} < k\right) = \Phi(k) - \Phi(-k) \end{aligned}$$

e.g. $k = 1$: 68%

$k = 2$: 95%

$k = 3$: 99%

Halloween Brain Break



Normal Distribution



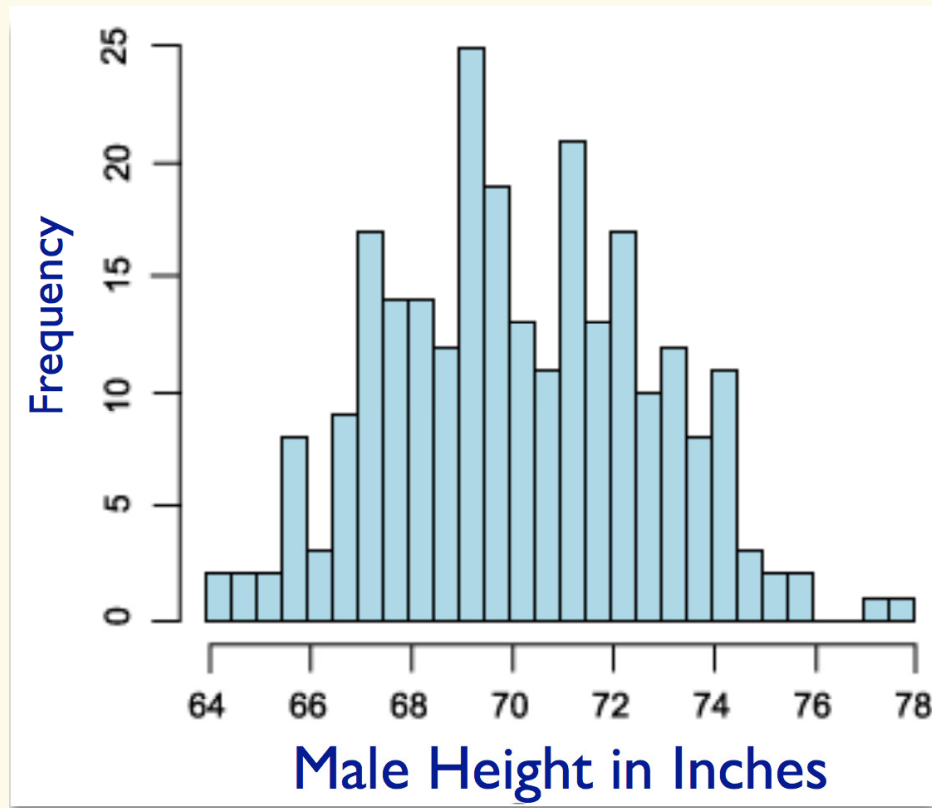
Paranormal Distribution

Agenda

- Normal Distribution
- Practice with Normals
- Central Limit Theorem (CLT) ◀

Gaussian in Nature

Empirical distribution of collected data often resembles a Gaussian ...



e.g. Height distribution resembles Gaussian.

R.A.Fisher (1918) observed that the height is likely the outcome of the sum of many independent random parameters, i.e., can written as

$$X = X_1 + \dots + X_n$$

Sum of Independent RVs

i.i.d. = independent and identically distributed

X_1, \dots, X_n i.i.d. with expectation μ and variance σ^2

Define

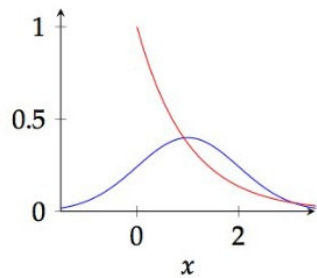
$$S_n = X_1 + \dots + X_n$$

$$\mathbb{E}[S_n] = \mathbb{E}[X_1] + \dots + \mathbb{E}[X_n] = n\mu$$

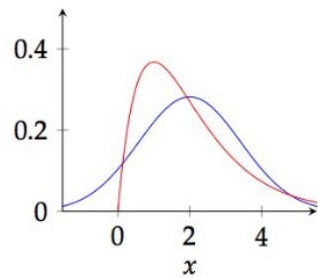
$$\text{Var}(S_n) = \text{Var}(X_1) + \dots + \text{Var}(X_n) = n\sigma^2$$

Empirical observation: S_n looks like a normal RV as n grows.

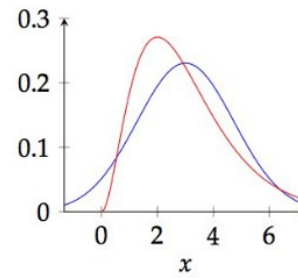
Example: Sum of n i.i.d. $\text{Exp}(1)$ random variables



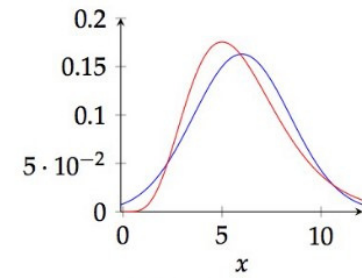
(a) $n = 1$



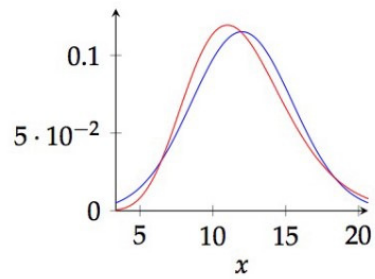
(b) $n = 2$



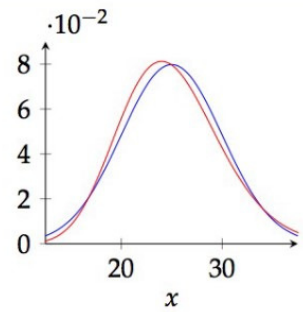
(c) $n = 3$



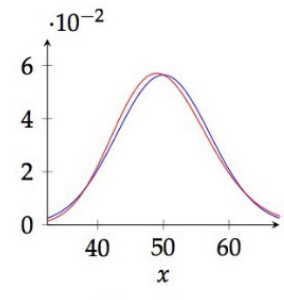
(d) $n = 6$



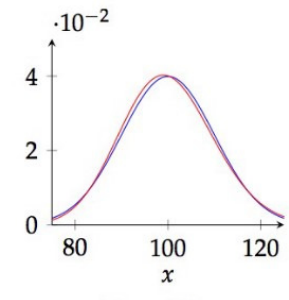
(e) $n = 12$



(f) $n = 25$

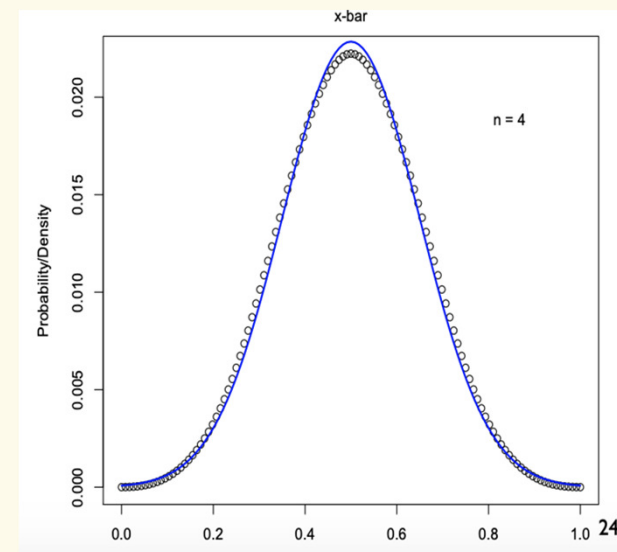
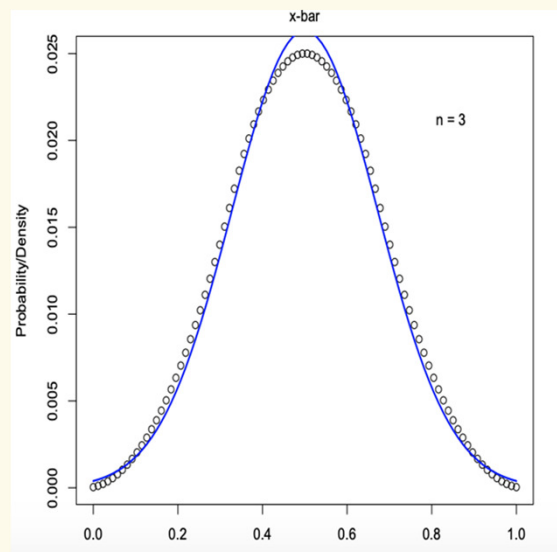
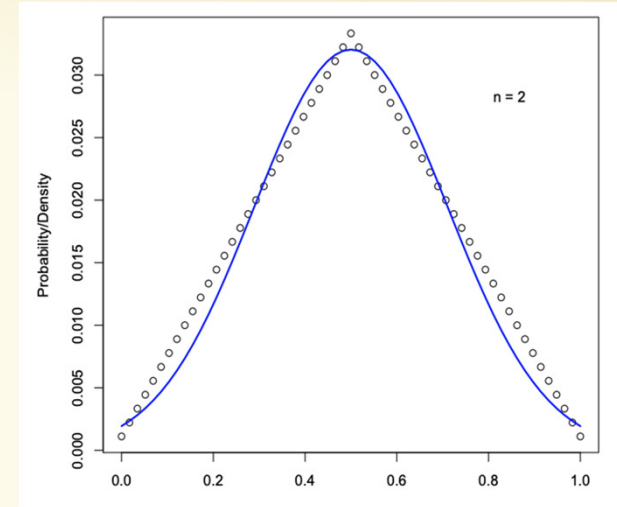
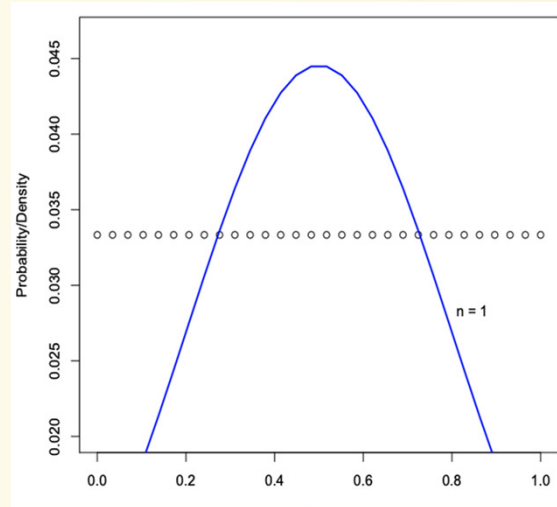


(g) $n = 50$

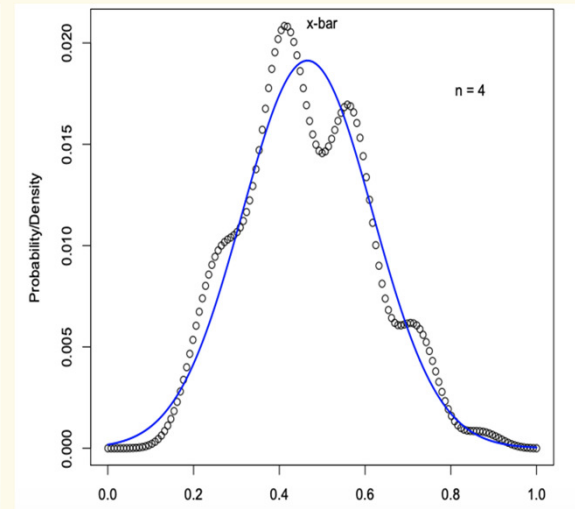
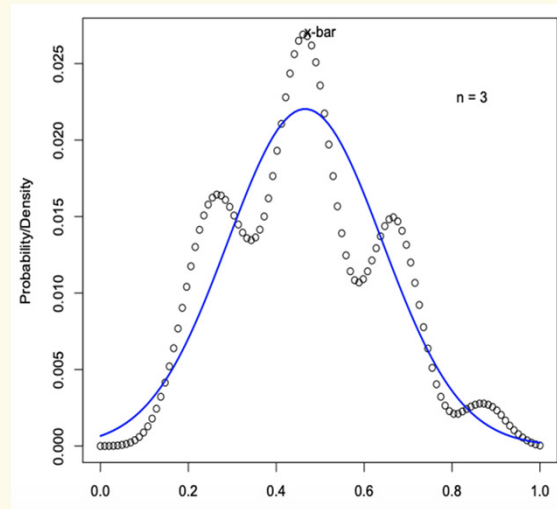
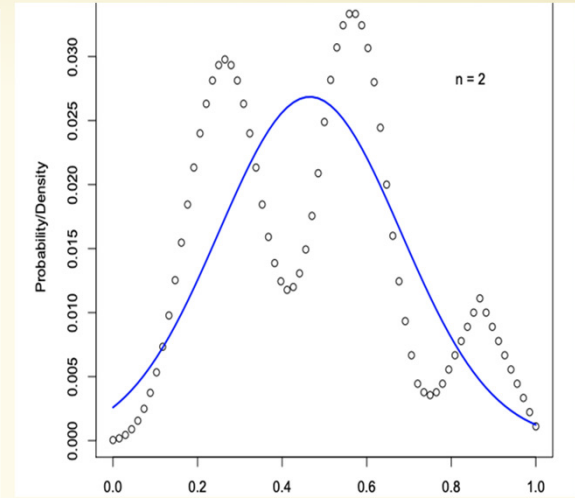
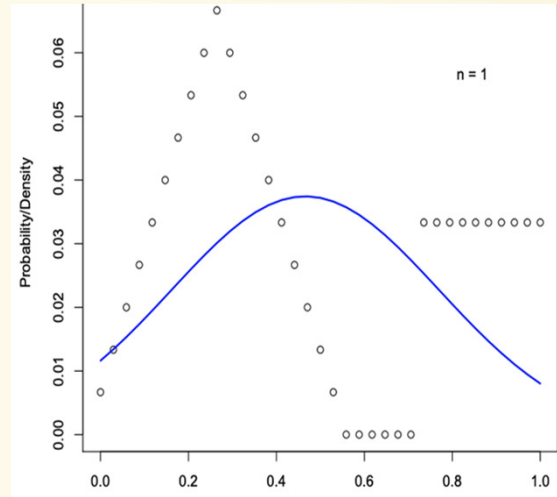


(h) $n = 100$

CLT (Idea)



CLT (Idea)



Central Limit Theorem

X_1, \dots, X_n i.i.d., each with expectation μ and variance σ^2

Define $S_n = X_1 + \dots + X_n$ and

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$$\mathbb{E}[Y_n] = \frac{1}{\sigma\sqrt{n}} (\mathbb{E}[S_n] - n\mu) = \frac{1}{\sigma\sqrt{n}} (n\mu - n\mu) = 0$$

$$\text{Var}(Y_n) = \frac{1}{\sigma^2 n} (\text{Var}(S_n - n\mu)) = \frac{\text{Var}(S_n)}{\sigma^2 n} = \frac{\sigma^2 n}{\sigma^2 n} = 1$$

Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

Theorem. (Central Limit Theorem) The CDF of Y_n converges to the CDF of the standard normal $\mathcal{N}(0,1)$, i.e.,

$$\lim_{n \rightarrow \infty} P(Y_n \leq y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-x^2/2} dx$$

Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

Theorem. (Central Limit Theorem) The CDF of Y_n converges to the CDF of the standard normal $\mathcal{N}(0,1)$, i.e.,

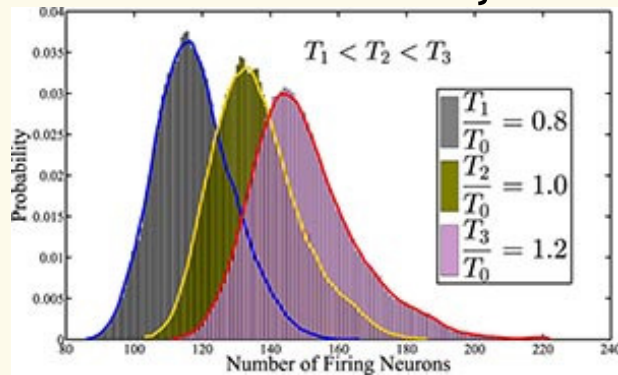
$$\lim_{n \rightarrow \infty} P(Y_n \leq y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-x^2/2} dx$$

Also stated as:

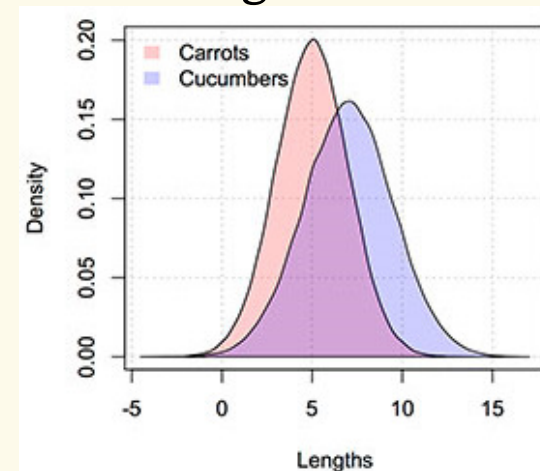
- $\lim_{n \rightarrow \infty} Y_n \rightarrow \mathcal{N}(0,1)$
- $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n X_i \rightarrow \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$ for $\mu = \mathbb{E}[X_i]$ and $\sigma^2 = \text{Var}(X_i)$

CLT → Normal Distribution EVERYWHERE

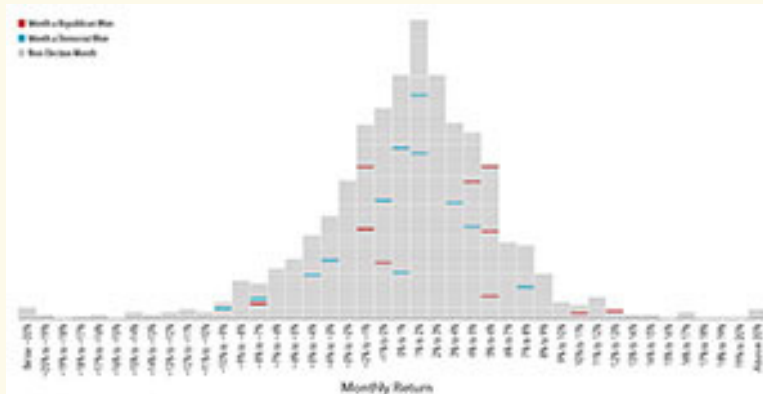
Neuron Activity



Vegetables



S&P 500 Returns after Elections



Examples from:
<https://galtonboard.com/probabilityexamplesinlife>