CSE 312

Foundations of Computing II

Lecture 3: More counting

Binomial Coefficients, Binomial Theorem, Inclusion-Exclusion

Announcements

- Office hours start today
  - In particular, I will be available for office hours starting right after class today (CSE 668)
- Problem Set 1
  - Read the first page for how to write up your homework solutions. Don’t wait until you are working on the questions to figure it out.
  - Section solutions are another good place to look at for examples.
- Resources
  - Textbook readings can provide another perspective
  - Theorems & Definitions sheet – https://www.alextsun.com/files/defs_thms.pdf
  - Office Hours
  - EdStem discussion
- EdStem discussion etiquette
  - OK to publicly discuss content of the course and any confusion over topics discussed in class, but **not *solutions*** for current homework problems, or anything about current exams that have not yet been graded.
  - It is also acceptable to ask for clarifications about what current homework problems are asking and concepts behind them, just not about their solutions.

Recap of Last Time

Permutations. The number of orderings of n distinct objects

$$n! = n \times (n-1) \times \cdots \times 2 \times 1$$

Example: How many sequences in $\{1,2,3\}^3$ with no repeating elements?

k-Permutations. The number of orderings of **only** k out of n distinct objects

$$P(n,k)$$

$$= n \times (n-1) \times \cdots \times (n-k+1)$$

$$= \frac{n!}{(n-k!)}$$

Example: How many sequences of 5 distinct alphabet letters from $\{A, B, \dots, Z\}$?

Combinations / Binomial Coefficient. The number of ways to select k out of n objects, where ordering of the selected k does not matter:

$$\binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!\,(n-k)!}$$

*Example: How many size-5 **subsets** of $\{A, B, \dots, Z\}$?*

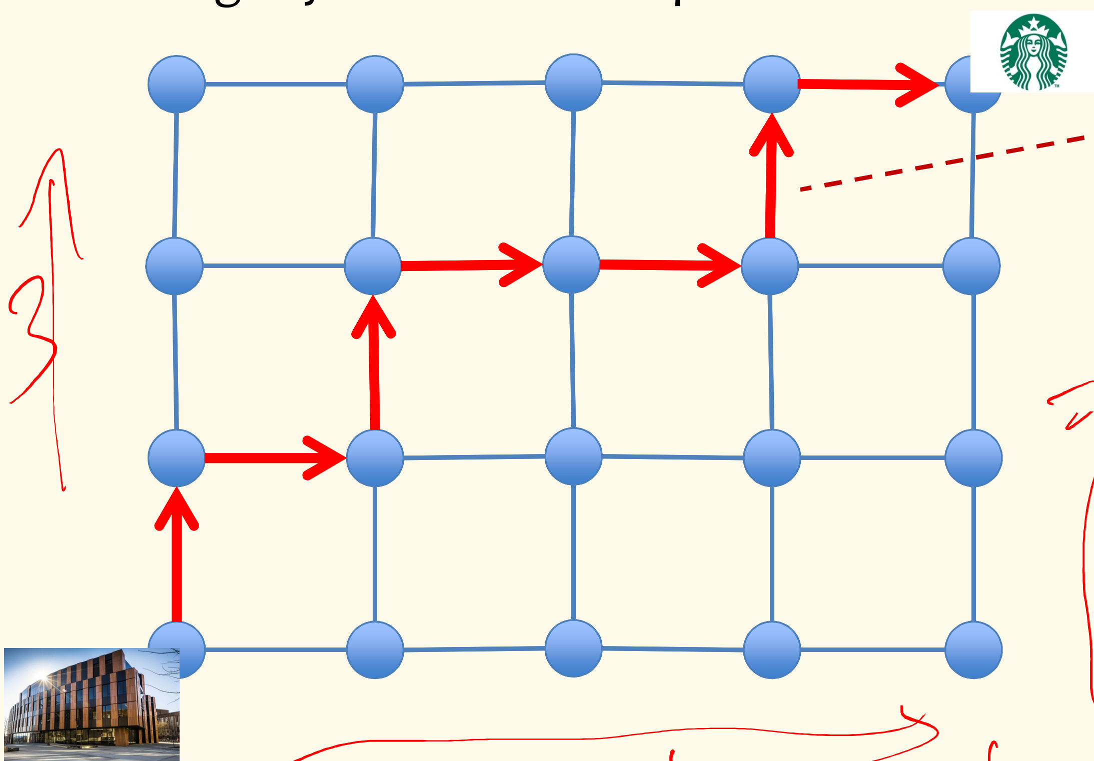
Example: How many shortest paths from Gates to Starbucks?

Example: How many solutions $(x_1, \dots, x_k)$ such that $x_1, \dots, x_k \geq 0$ and $\sum_{i=1}^{k} x_i = n$?

Recap* Example – Counting Paths

$$\text{Path} \in \{\uparrow, \rightarrow\}^7$$

A slightly modified example



Example path:

$(\uparrow, \rightarrow, \uparrow, \rightarrow, \rightarrow, \uparrow, \rightarrow)$

$\# = \binom{7}{4} = \binom{7}{3}$

also like SEATTLE Example

Imagine $\uparrow_1\ \uparrow_2\ \uparrow_3\ \rightarrow_1\ \rightarrow_2\ \rightarrow_3\ \rightarrow_4$

order matters

3

4

$= \dfrac{7!}{3!\,4!}$

order of ↑ doesn't matter

order of → doesn't matter

Agenda

- Binomial Coefficients
- Binomial Theorem
- Inclusion-Exclusion

Binomial Coefficient – Many interesting and useful properties

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad \binom{n}{n} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{0} = 1$$

Fact. $\binom{n}{k} = \binom{n}{n-k}$ **Symmetry in Binomial Coefficients**

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ **Pascal's Identity**

Fact. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n}{n-k}$

This is called an Algebraic proof, i.e., Prove by checking algebra

Proof. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$

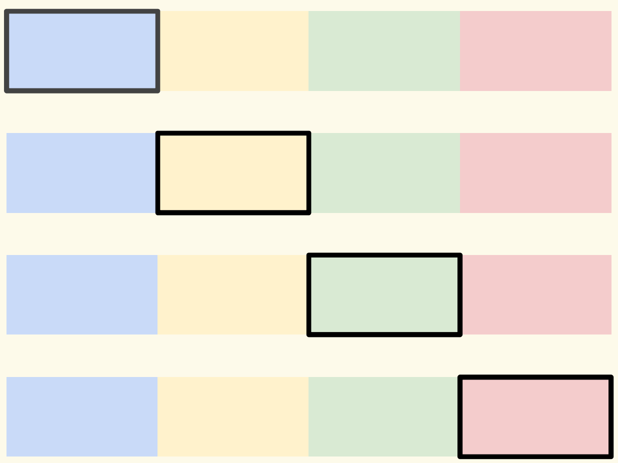
Why??

Symmetry in Binomial Coefficients – A different proof

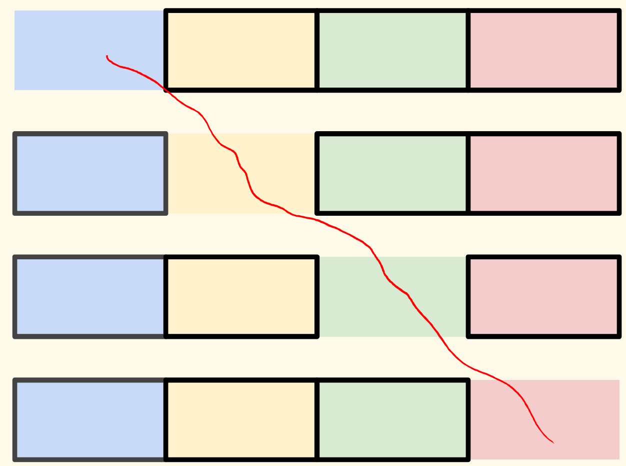
Fact. $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

1. Choose which k elements are included
2. Choose which $n-k$ elements are excluded



$$\binom{4}{1} = 4 = \binom{4}{3}$$



Symmetry in Binomial Coefficients – A different proof

Fact. $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

1. Choose which k elements are included
2. Choose which $n - k$ elements are excluded

Format for a combinatorial argument/proof of $a = b$

- Let S be a set of objects
- Show how to count $|S|$ one way $\Rightarrow$ $|S| = a$
- Show how to count $|S|$ another way $\Rightarrow$ $|S| = b$

Combinatorial argument/proof

- Elegant
- Simple
- Intuitive

Algebraic argument

- Brute force
- Less Intuitive

Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

How to prove Pascal's identity?

Algebraic argument:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$
$$= 20\ years\ later\ \ldots$$
$$= \frac{n!}{k!\,(n-k)!}$$
$$= \binom{n}{k}$$

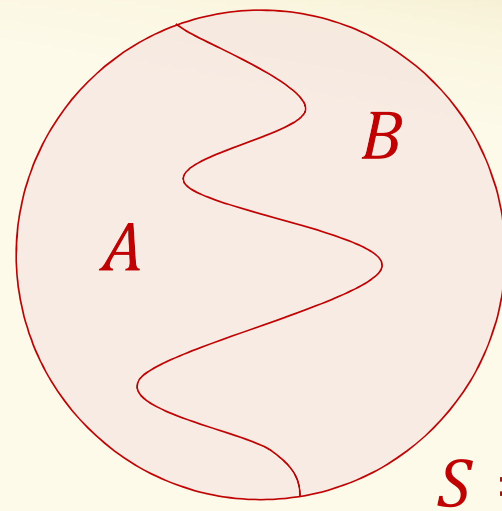
Hard work and not intuitive

Let's see a combinatorial argument

Example – Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$|S| \quad = \quad |A| \quad + \quad |B|$$



$S = A \cup B$

Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have the sizes above.
- The equation then follows by the Sum Rule.

Example – Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| \quad = \quad |A| \quad + \quad |B|$

Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have these sizes

$|S| = \binom{n}{k}$

S: set of size k subsets of $[n] = \{1, 2, \cdots, n\}$.

e.g. $n = 4, k = 2,$ $S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

A: set of size k subsets of $[n]$ that DO include n

$$A = \{\{1,4\}, \{2,4\}, \{3,4\}\}$$

B: set of size k subsets of $[n]$ that DON'T include n

$$B = \{\{1,2\}, \{1,3\}, \{2,3\}\}$$

Example – Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$|S| = |A| + |B|$$

Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have these sizes

S: set of size k subsets of $[n] = \{1, 2, \cdots, n\}$.

n is in set, need to choose other $k-1$ elements from $[n-1]$

$$|A| = \binom{n-1}{k-1}$$

A: set of size k subsets of $[n]$ that DO include n

$\{ (1,4) \ (2,4) \ (3,4) \}$

B: set of size k subsets of $[n]$ that DON'T include n

n not in set, need to choose k elements from $[n-1]$

$$|B| = \binom{n-1}{k}$$

Agenda

- Binomial Coefficients
- Binomial Theorem
- Inclusion-Exclusion

Binomial Theorem: Idea

$$(x+y)^2 = (x+y)(x+y)$$
$$= xx + xy + yx + yy$$
$$= x^2 + 2xy + y^2$$

Poll: What is the coefficient for xy^3?

All ok

A. 4

B. $\binom{4}{1}$

C. $\binom{4}{3}$

D. 3

https://pollev.com/paulbeame028

$$(x+y)^4 = (x+y)(x+y)\,(x+y)\,(x+y)$$

$$= xxxx + yyyy + xyxy + yxyy + \ \ldots$$

Binomial Theorem: Idea

$$(x+y)^n = \overbrace{(x+y)\ldots(x+y)}^{n \text{ copies}}$$

Each term is of the form $x^k y^{n-k}$, since each term is made by multiplying exactly n variables, either x or y, one from each copy of $(x+y)$

How many times do we get $x^k y^{n-k}$?

The number of ways to choose x from exactly k of the n copies of $(x+y)$ (the other $n-k$ choices will be y) which is:

$$\binom{n}{k} = \binom{n}{n-k}$$

Binomial Theorem

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Many properties of sums of binomial coefficients can be found by plugging in different values of x and y in the Binomial Theorem.

$x=y=1$ $\Rightarrow$

Corollary.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Brain Break

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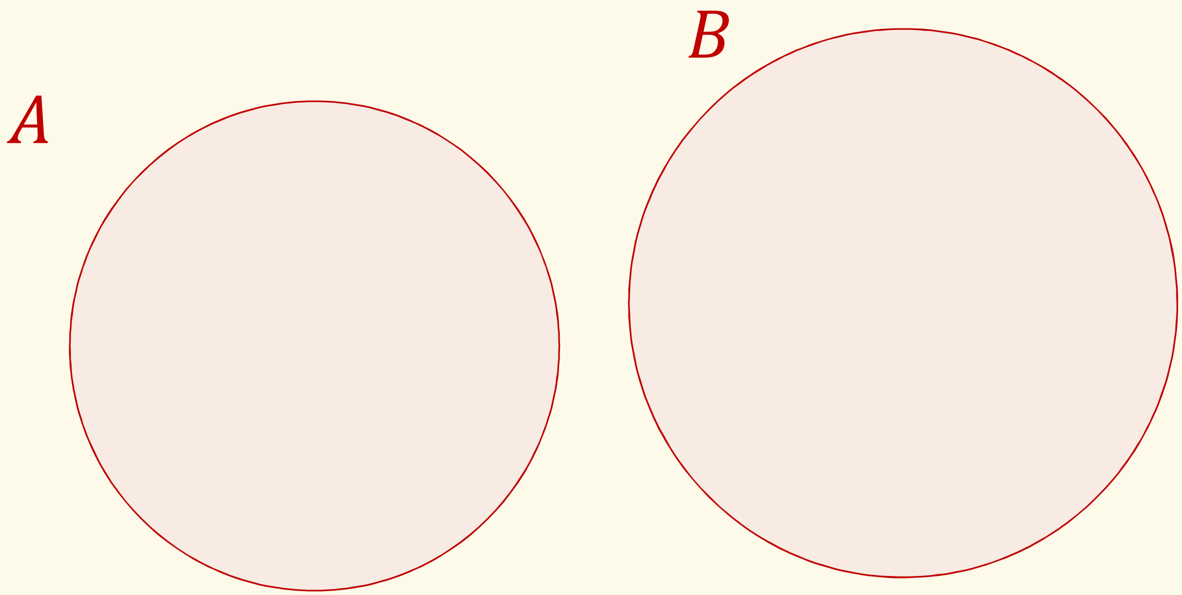
Recap Disjoint Sets

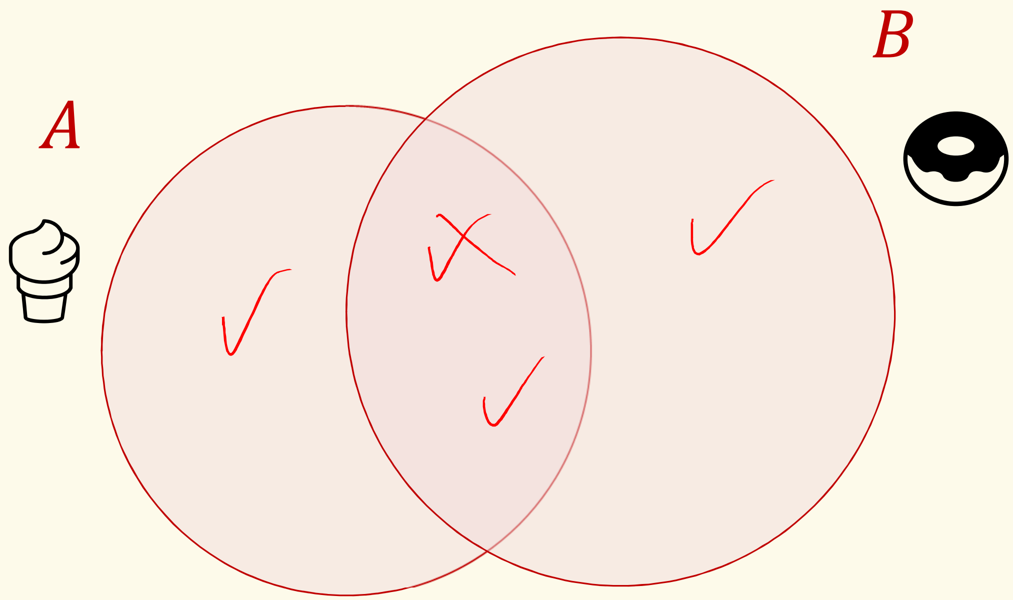
Sets that do not contain common elements ($A \cap B = \emptyset$)



Sum Rule: $|A \cup B| = |A| + |B|$

Inclusion-Exclusion

But what if the sets are not disjoint?



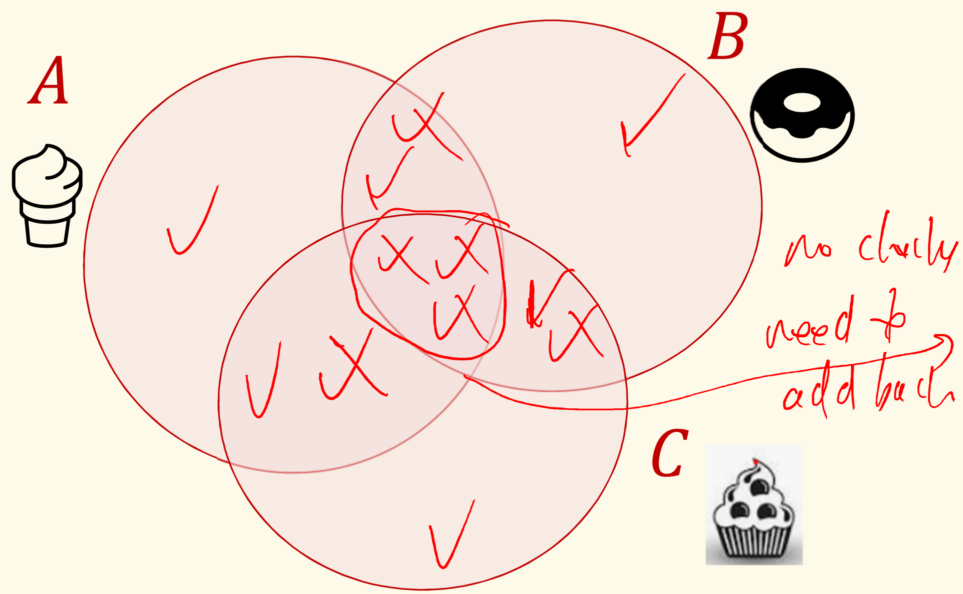
$|A| = 43$
$|B| = 20$
$|A \cap B| = 7$
$|A \cup B| = ???$

Fact. $|A \cup B| = |A| + |B| - |A \cap B|$

Inclusion-Exclusion

Not drawn to scale

What if there are three sets?



$|A| = 43$

$|B| = 20$

$|C| = 35$

$|A \cap B\ | = 7$

$|A \cap C\ | = 16$

$|B \cap C\ | = 11$

$|A \cap B \cap C\ | = 4$

$|A \cup B \cup C| = ???$

Fact.

$$|A \cup B \cup C| = |A| + |B| + |C| \\ - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Inclusion-Exclusion

Let A, B be sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

In general, if $A_1, A_2, \dots, A_n$ are sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = singles - doubles + triples - quads + \dots$$
$$= (|A_1| + \cdots + |A_n|) - (|A_1 \cap A_2| + \dots + |A_{n-1} \cap A_n|) + \dots$$

Example: RSA

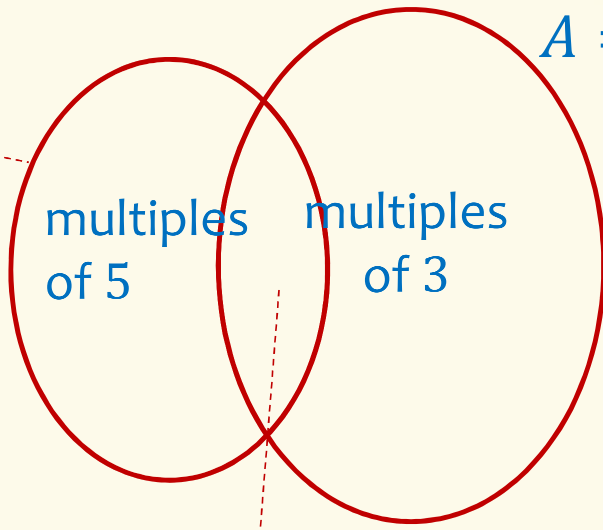
- In encrypting messages using RSA one starts with
  - Two big prime numbers p and q that are kept secret
  - Encodes messages using arithmetic $\mathrm{mod}\ N$ for $N = pq$.
  - One needs to work with numbers $\mathrm{mod}\ N$ that have no common factors with N ("co-prime with N")
    - Otherwise the secret leaks or decryption may not be defined uniquely.
  - To define RSA one needs to know how many such numbers there are…

Example: $p = 3, q = 5 \quad \text{mod } 15 = 3 * 5$

$B = \{0, 5, 10\}$

$A = \{0, 3, 6, 9, 12\}$

$|B| = 3$



$|A| = 5$

$A \cap B$ contains multiples of 3 & 5 $(\text{mod } 15)$ $\quad A \cap B = \{0\}$

\# Integers between 0 and 14 that share a non-trivial divisor with 15= $|A| + |B| - |A \cap B| = 3 + 5 - 1 = 7$

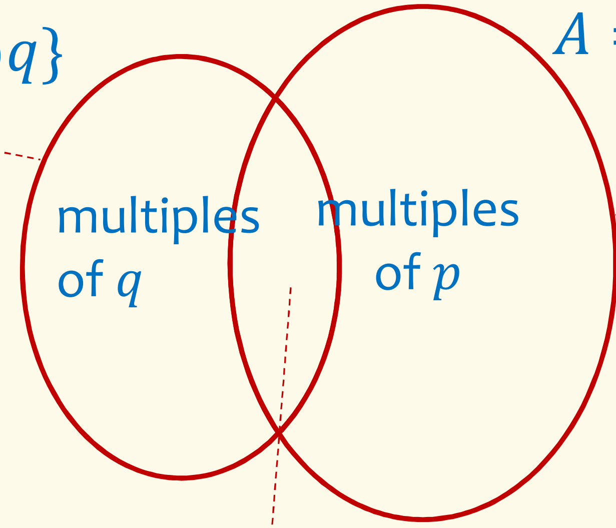
\# Integers between 0 and $N - 1$ that are co-prime with N
= $15 - 7 = 8$ $= 2 \cdot 4$

Integers mod N co-prime with $N = pq$ for p, q prime

$B = \{0, q, 2q, \ldots, (p-1)q\}$

$A = \{0, p, 2p, \ldots, (q-1)p\}$

$|B| = p$



$|A| = q$

$A \cap B$ contains multiples of p & $q \pmod{N}$ $\quad A \cap B = \{0\}$

Integers between 0 and $N-1$ that share a non-trivial divisor with N
= $|A| + |B| - |A \cap B| = p + q - 1$

Integers between 0 and $N-1$ that are co-prime with N
= $N - (p + q - 1) = pq - p - q + 1 = (p-1)(q-1)$

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