CSE 312

Foundations of Computing II

Lecture 10: Variance and Independence of RVs

Recap Linearity of Expectation

Theorem. For any two random variables X and Y *(X, Y do not need to be independent)*

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

Theorem. For any random variables $X_1, \dots, X_n$, and real numbers $a_1, \dots, a_n \in \mathbb{R}$,

$$\mathbb{E}[a_1 X_1 + \cdots + a_n X_n] = a_1 \mathbb{E}[X_1] + \cdots + a_n \mathbb{E}[X_n].$$

For any event A, can define the indicator random variable X for A

$$X_A = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{if event } A \text{ does not occur} \end{cases}$$

$$P(X_A = 1) = P(A)$$
$$P(X_A = 0) = 1 - P(A)$$

$\mathbb{E}[X_A] = 1 \cdot P(A) = P(A)$

Recap Linearity is special!

In general $\mathbb{E}[g(X)] \neq g\big(\mathbb{E}(X)\big)$

E.g., $X = \begin{cases} +1 \text{ with prob } 1/2 \\ -1 \text{ with prob } 1/2 \end{cases}$

Then: $\mathbb{E}[X^2] \neq \mathbb{E}[X]^2$

How DO we compute $\mathbb{E}[g(X)]$?

Recap Expected Value of $g(X)$

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of $Y = g(X)$ is

$$\mathbb{E}[Y] = \sum_{\omega \in \Omega} g\big(X(\omega)\big) \cdot P(\omega)$$

or equivalently

$$\mathbb{E}[Y] = \sum_{x \in X(\Omega)} g(x) \cdot P(X = x) \qquad = \sum_{x \in \Omega_X} g(x) \cdot p_X(x)$$

Also known as **LOTUS:** "Law of the unconscious statistician

(nothing special going on in the discrete case)

Example: Expectation of $g(X)$

Suppose we rolled a fair, 6-sided die in a game.

- Let X be the result of the dice roll.

$i \in \{1, \ldots, 6\}$

$P(X = i) = \frac{1}{6}$

You will win the cube of the number rolled in dollars, times 10.

What are your expected winnings?

$g(x) = 10x^3$

$$\mathbb{E}[10X^3] = \sum_{x \in X(\Omega)} g(x) \cdot P(X = x)$$

$10 \, \mathbb{E}[X^3]$ $\quad \{1, \ldots, 6\}$ $\quad 10x^3$ $\quad \frac{1}{6}$

$$10 \sum_{k=1}^{6} k^3 \cdot \frac{1}{6}$$

Agenda

- Variance
- Properties of Variance
- Independent Random Variables
- Properties of Independent Random Variables

Expected Value of X = # of heads

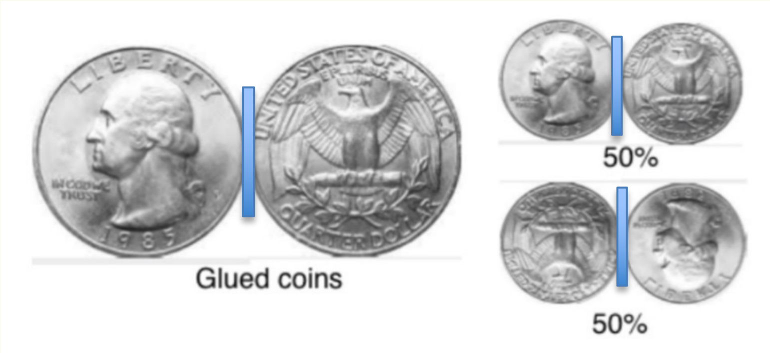
Each coin shows up heads half the time, but very different joint behaviors!

Two fair coins

$P(HT) = P(TH) = 0.25$
$P(HH) = P(TT) = 0.25$

$$\mathbb{E}[X] = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

Glued coins



$P(HT) = P(TH) = 0.5$
$P(HH) = P(TT) = 0$

$$\mathbb{E}[X] = 1 \cdot 1 = 1$$

Attached coins

$P(HH) = P(TT) = 0.4$
$P(HT) = P(TH) = 0.1$

$$\mathbb{E}[X] = 1 \cdot 0.2 + 2 \cdot 0.4 = 1$$

Two Games

$2 \cdot \frac{1}{3} - 1 \cdot \frac{2}{3} = 0$

Game 1: *In every round, you win \$2 with probability 1/3, lose \$1 with probability 2/3.*

W_1 = payoff in a round of Game 1

$$P(W_1 = 2) = \frac{1}{3}, P(W_1 = -1) = \frac{2}{3}$$

$$\mathbb{E}[W_1] = 0$$

Game 2: *In every round, you win \$10 with probability 1/3, lose \$5 with probability 2/3.*

W_2 = payoff in a round of Game 2

$$P(W_2 = 10) = \frac{1}{3}, P(W_2 = -5) = \frac{2}{3}$$

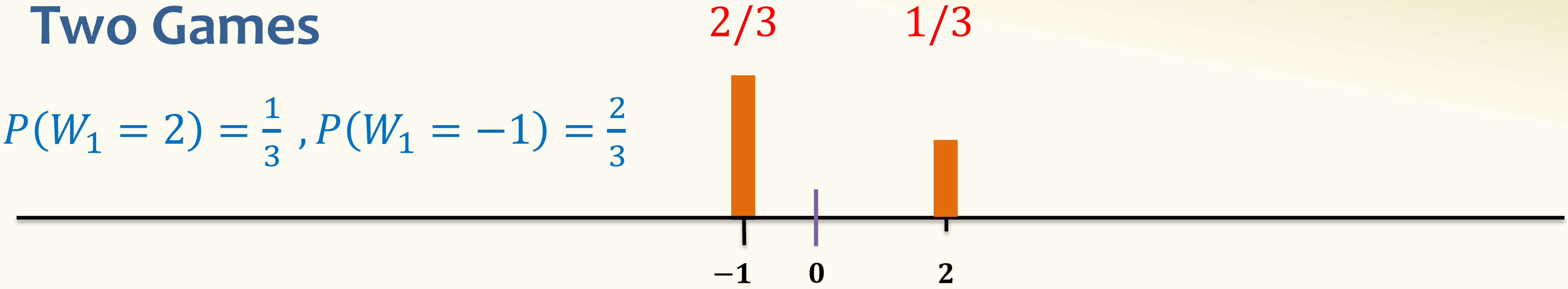
$$\mathbb{E}[W_2] = 0$$

Which game would you rather play?

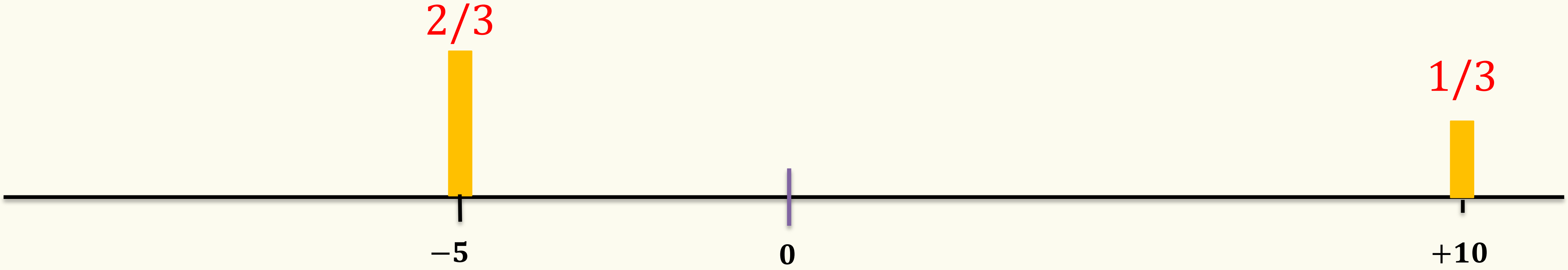
Somehow, Game 2 has higher volatility / exposure!

Two Games

$$P(W_1 = 2) = \frac{1}{3}\ , P(W_1 = -1) = \frac{2}{3}$$



$$P(W_2 = 10) = \frac{1}{3}\ , P(W_2 = -5) = \frac{2}{3}$$



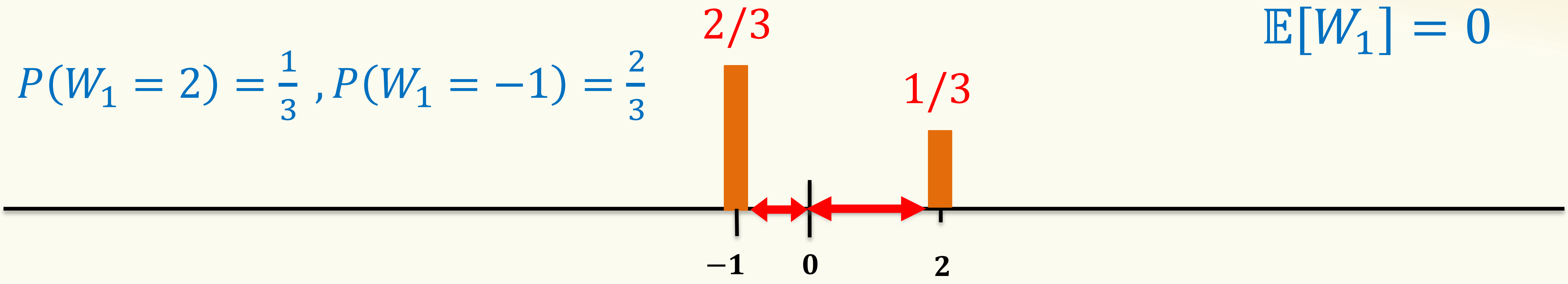
Same expectation, but clearly a very different distribution.

We want to capture the difference – New concept: **Variance**

Variance (Intuition, First Try)

$P(W_1 = 2) = \frac{1}{3}, P(W_1 = -1) = \frac{2}{3}$

$\mathbb{E}[W_1] = 0$



New quantity (random variable): How far from the expectation?

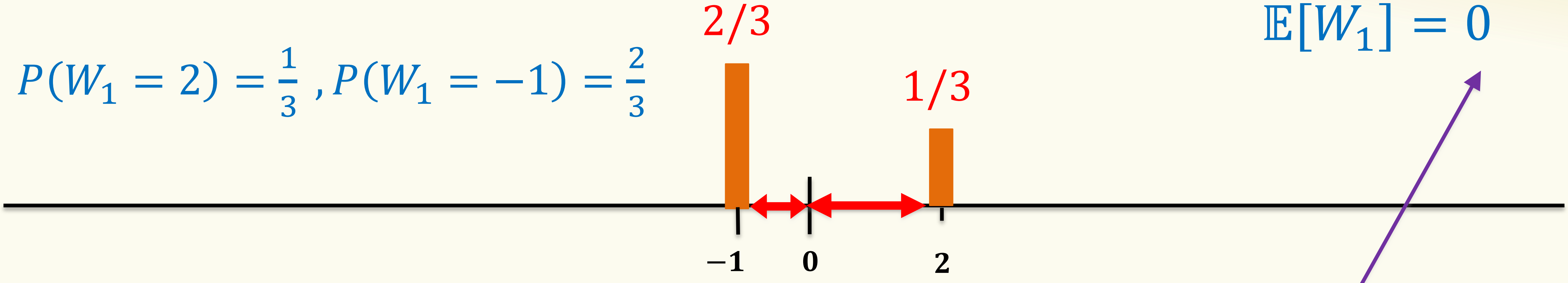
$\Delta(W_1) = W_1 - \mathbb{E}[W_1]$

$$\begin{aligned}\mathbb{E}[\Delta(W_1)] &= \mathbb{E}[W_1 - \mathbb{E}[W_1]] \\ &= \mathbb{E}[W_1] - \mathbb{E}[\mathbb{E}[W_1]] \\ &= \mathbb{E}[W_1] - \mathbb{E}[W_1] \\ &= 0\end{aligned}$$

Variance (Intuition, Better Try)

$P(W_1 = 2) = \frac{1}{3}$, $P(W_1 = -1) = \frac{2}{3}$

$\mathbb{E}[W_1] = 0$



A better quantity (random variable): How far from the expectation?

$\Delta(W_1) = (W_1 - \mathbb{E}[W_1])^2$

$P(\Delta(W_1) = 1) = \frac{2}{3}$

$P(\Delta(W_1) = 4) = \frac{1}{3}$

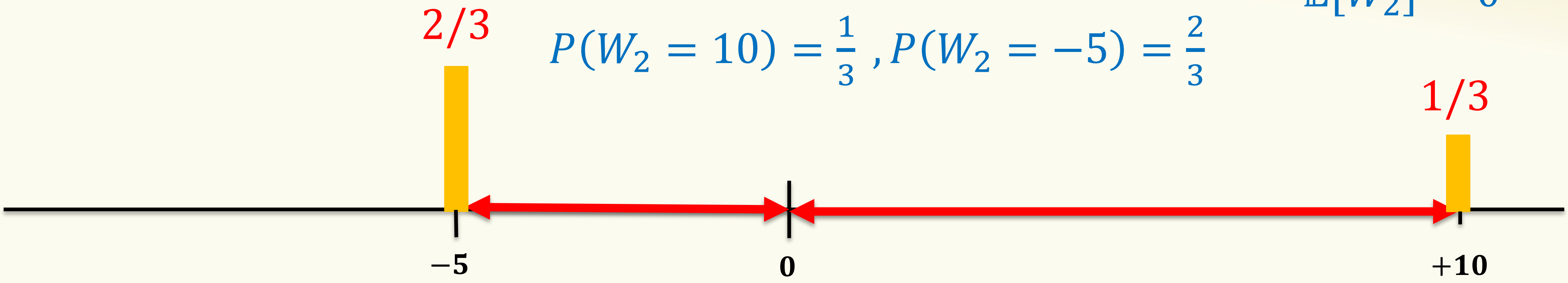
$$\mathbb{E}[\Delta(W_1)] = \mathbb{E}[(W_1 - \mathbb{E}[W_1])^2] = \mathbb{E}[u_1^2]$$

$$= \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 4$$

$$= 2$$

$g(x) = x^2$

Variance (Intuition, Better Try)



$P(W_2 = 10) = \frac{1}{3}, P(W_2 = -5) = \frac{2}{3}$

$\mathbb{E}[W_2] = 0$

A better quantity (random variable): How far from the expectation?

$$\Delta(W_2) = (W_2 - \mathbb{E}[W_2])^2$$

$$\mathbb{P}(\Delta(W_2) = 25) = \frac{2}{3}$$

$$\mathbb{P}(\Delta(W_2) = 100) = \frac{1}{3}$$

$$\mathbb{E}[\Delta(W_2)] = \mathbb{E}[(W_2 - \mathbb{E}[W_2])^2]$$

$$= \frac{2}{3} \cdot 25 + \frac{1}{3} \cdot 100$$

$$= 50$$

Poll:
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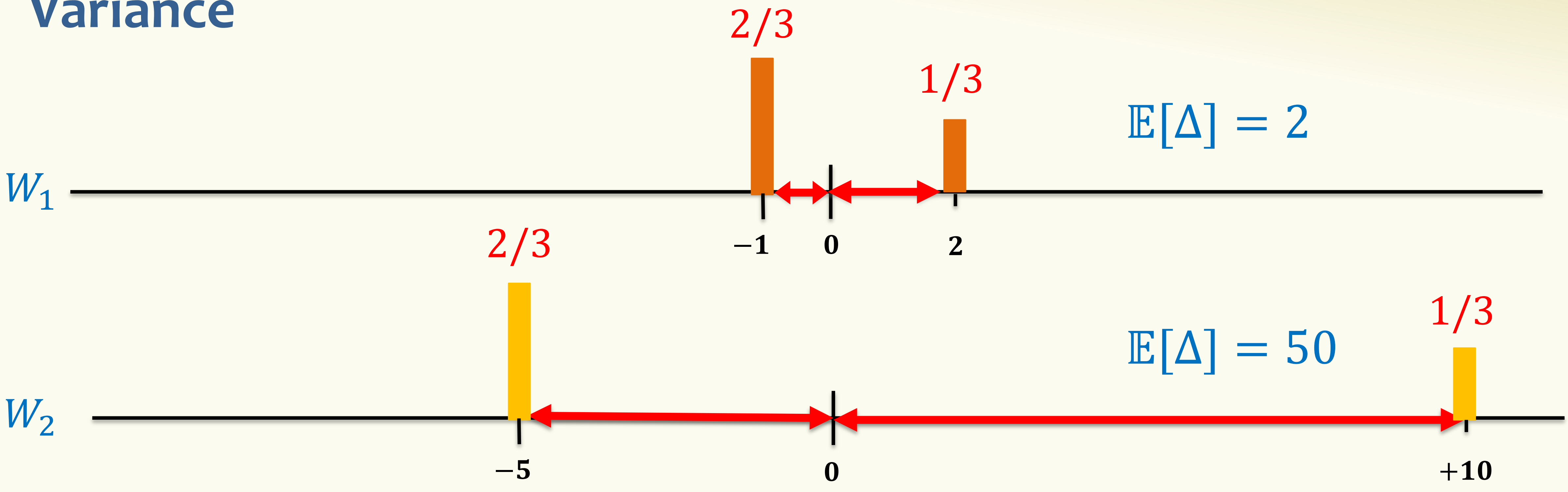
A. 0
B. 20/3
C. 50
D. 2500

Variance



We say that W_2 has **"higher variance"** than W_1.

Variance

Definition. The **variance** of a (discrete) RV X is

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_x p_X(x) \cdot (x - \mathbb{E}[X])^2$$

Recall $\mathbb{E}[X]$ is a **constant**, not a random variable itself.

Standard deviation: $\sigma(X) = \sqrt{\mathrm{Var}(X)}$

Intuition: Variance (or standard deviation) is a quantity that measures, in expectation, how "far" the random variable is from its expectation.

Variance – Example 1

X fair die

- $P(X = 1) = \cdots = P(X = 6) = 1/6$
- $\mathbb{E}[X] = 3.5$

$\mathrm{Var}(\mathrm{X}) = ?$ $\left(x - \mathbb{E}(X)\right)^2$

Variance – Example 1

X fair die

- $P(X = 1) = \cdots = P(X = 6) = 1/6$
- $\mathbb{E}[X] = 3.5$

$$\mathrm{Var}(X) = \sum_x P(X = x) \cdot (x - \mathbb{E}[X])^2$$

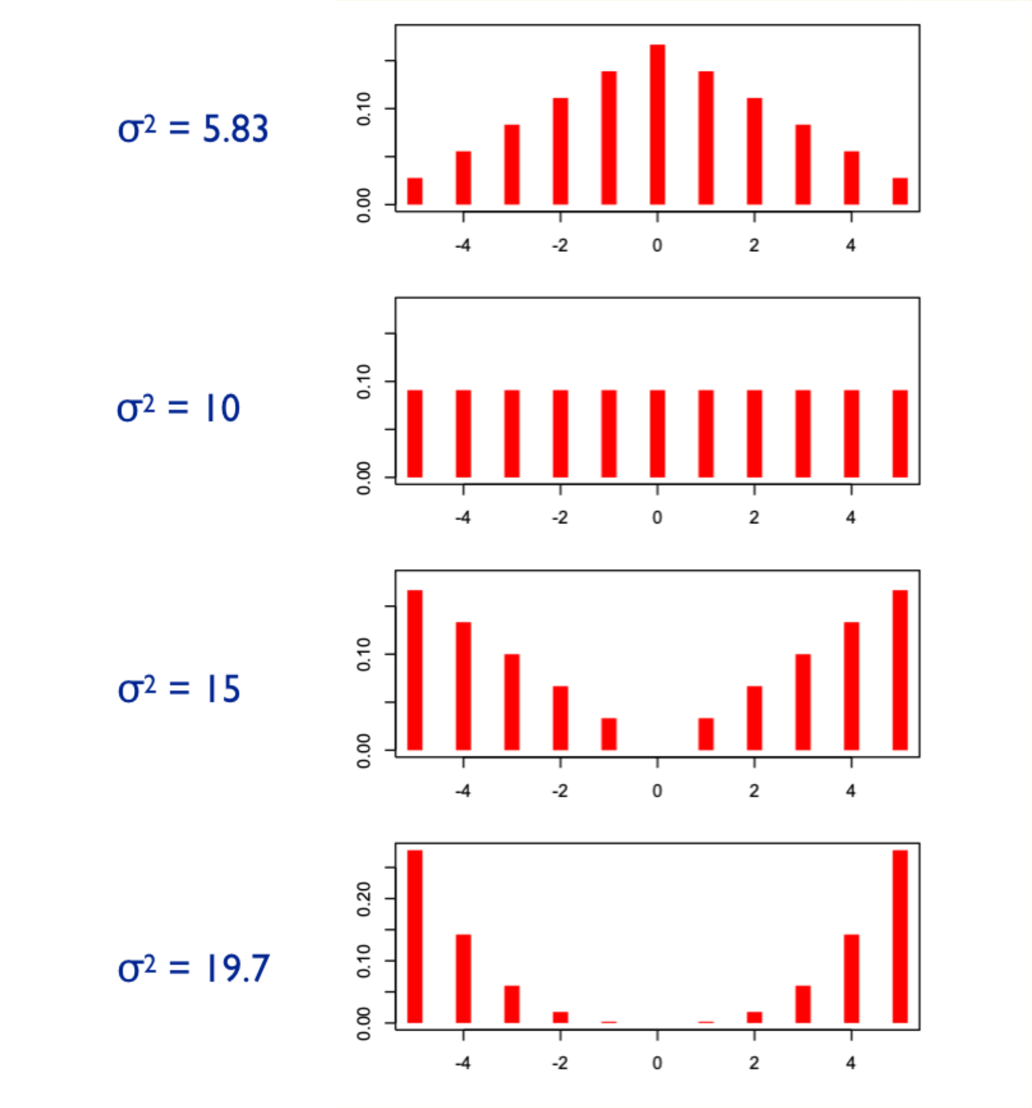
$$= \frac{1}{6}[(1 - 3.5)^2 + (2 - 3.5)^2 + (3 - 3.5)^2 + (4 - 3.5)^2 + (5 - 3.5)^2 + (6 - 3.5)^2]$$

$$= \frac{2}{6}[2.5^2 + 1.5^2 + 0.5^2] = \frac{2}{6}\left[\frac{25}{4} + \frac{9}{4} + \frac{1}{4}\right] = \frac{35}{12} \approx 2.91677 \ldots$$

Variance in Pictures

Captures how much "spread' there is in a pmf

All pmfs have same expectation



Agenda

- Variance
- Properties of Variance
- Independent Random Variables
- Properties of Independent Random Variables

Variance – Properties

Definition. The **variance** of a (discrete) RV X is

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_x p_X(x) \cdot (x - \mathbb{E}[X])^2$$

Theorem. For any $a, b \in \mathbb{R}$, $\mathrm{Var}(a \cdot X + b) = a^2 \cdot \mathrm{Var}(X)$

(Proof: Exercise!)

Theorem. $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

Variance

Theorem. $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

Proof: $\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$

Recall $\mathbb{E}[X]$ is a **constant**

$$= \mathbb{E}[X^2 - 2\mathbb{E}[X] \cdot X + \mathbb{E}[X]^2]$$

$$= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[X]^2$$

(linearity of expectation!)

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$\mathbb{E}[X^2]$ and $\mathbb{E}[X]^2$ are different !

Variance – Example 1

X fair die

- $\mathbb{P}(X = 1) = \cdots = \mathbb{P}(X = 6) = 1/6$
- $\mathbb{E}[X] = \frac{21}{6}$
- $\mathbb{E}[X^2] = \frac{91}{6}$

$$\mathrm{Var}(\mathrm{X}) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{105}{36} \approx 2.91677$$

Variance of Indicator Random Variables

Suppose that X_A is an indicator RV for event A with $P(A) = p$ so

$$\mathbb{E}[X_A] = P(A) = p$$

Since X_A only takes on values 0 and 1, we always have $X_A^2 = X_A$ so

$$\mathrm{Var}(X_A) = \mathbb{E}[X_A^2] - \mathbb{E}[X_A]^2 = \mathbb{E}[X_A] - \mathbb{E}[X_A]^2 = p - p^2 = p(1-p)$$

In General, $\mathrm{Var}(X + Y) \neq \mathrm{Var}(X) + \mathrm{Var}(Y)$

Proof by counter-example:

- Let X be a r.v. with pmf $P(X = 1) = P(X = -1) = 1/2$
  - What is $\mathbb{E}[X]$ and $\mathrm{Var}(X)$?
- Let $Y = -X$
  - What is $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$?

What is $\mathrm{Var}(X + Y)$?

Brain Break

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Random Variables and Independence

Comma is shorthand for AND

Definition. Two random variables X, Y are **(mutually) independent** if for all x, y,

$\{X = x\} \cap \{Y = y\}$

$$P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$$

Intuition: Knowing X doesn't help you guess Y and vice versa

Definition. The random variables $X_1, \ldots, X_n$ are **(mutually) independent** if for all $x_1, \ldots, x_n$,

$$P(X_1 = x_1, \ldots, X_n = x_n) = P(X_1 = x_1) \cdots P(X_n = x_n)$$

Note: No need to check for all subsets, but need to check for all outcomes!

Example

Let X be the number of heads in n *independent* coin flips of the same coin. Let $Y = X \bmod 2$ be the parity (even/odd) of X.

Are X and Y independent?

$$P(X = x, Y = c) \stackrel{?}{=} P(X = x) \cdot P(Y = c)$$

$x = 7$? $c = 0$ → 0; $P(X=x) > 0$; $P(Y=c) > 0$

Poll:

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A. Yes

B. No

Example

Make $2n$ *independent* coin flips of the same coin.

Let X be the number of heads in the first n flips and Y be the number of heads in the last n flips.

Are X and Y independent?

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A. Yes
B. No

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Important Facts about Independent Random Variables

Theorem. If X, Y independent, $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Theorem. If X, Y independent, $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Corollary. If $X_1, X_2, \dots, X_n$ mutually independent,

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i}^{n} \mathrm{Var}(X_i)$$

(Not Covered) Proof of $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Theorem. If X, Y independent, $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Proof

Let $x_i, y_i, i = 1, 2, \ldots$ be the possible values of X, Y.

$$\mathbb{E}[X \cdot Y] = \sum_i \sum_j x_i \cdot y_j \cdot P(X = x_i \wedge Y = y_j)$$

independence

$$= \sum_i \sum_j x_i \cdot y_i \cdot P(X = x_i) \cdot P(Y = y_j)$$

$$= \sum_i x_i \cdot P(X = x_i) \cdot \left(\sum_j y_j \cdot P(Y = y_j) \right)$$

$$= \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

Note: *NOT* true in general; see earlier example $\mathbb{E}[X^2] \neq \mathbb{E}[X]^2$

(Not Covered) Proof of $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$

Theorem. If X, Y independent, $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$

Proof

$$Var(X+Y)$$

$$= \mathbb{E}[(X+Y)^2] - (\mathbb{E}[X+Y])^2$$

$$= \mathbb{E}[X^2 + 2XY + Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2$$

$$= \mathbb{E}[X^2] + 2\,\mathbb{E}[XY] + \mathbb{E}[Y^2] - (\mathbb{E}[X]^2 + 2\,\mathbb{E}[X]\,\mathbb{E}[Y] + \mathbb{E}[Y]^2)$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2 + \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 + 2\,\mathbb{E}[XY] - 2\,\mathbb{E}[X]\,\mathbb{E}[Y]$$

$$= Var(X) + Var(Y) + 2\,\mathbb{E}[XY] - 2\,\mathbb{E}[X]\,\mathbb{E}[Y]$$

$$= Var(X) + Var(Y)$$

linearity

equal by independence

Example – Coin Tosses

We flip n *independent* coins, each one heads with probability p

- $X_i = \begin{cases} 1, & i^{\text{th}} \text{ outcome is heads} \\ 0, & i^{\text{th}} \text{ outcome is tails.} \end{cases}$
- $Z =$ number of heads

Fact. $Z = \sum_{i=1}^{n} X_i$

$P(X_i = 1) = p$
$P(X_i = 0) = 1 - p$

$P(Z = k) = \binom{n}{k} p^k (1-p)^{n-k}$

What is $\mathbb{E}[Z]$? What is $\mathrm{Var}(Z)$?

Note: $X_1, \ldots, X_n$ are <u>mutually</u> independent! [Verify it formally!]

$$\mathrm{Var}(Z) = \sum_{i=1}^{n} \mathrm{Var}(X_i) = n \cdot p(1-p)$$

Note $\mathrm{Var}(X_i) = p(1-p)$