CSE 312

# Foundations of Computing II

## Lecture 21: Chernoff Bound & Union Bound

## Review Tail Bounds

Putting a limit on the probability that a random variable is in the "tails" of the distribution (e.g., not near the middle).

Usually statements in the form of

$$P(X \geq a) \leq b$$

or

$$P(|X - \mathbb{E}[X]| \geq a) \leq b$$

## Review Markov's and Chebyshev's Inequalities

**Theorem (Markov's Inequality).** Let $X$ be a random variable taking only non-negative values. Then, for any $t > 0$,

$$P(X \geq t) \leq \frac{\mathbb{E}[X]}{t}.$$

**Theorem (Chebyshev's Inequality).** Let $X$ be a random variable. Then, for any $t > 0$,

$$P(|X - \mathbb{E}[X]| \geq t) \leq \frac{\mathrm{Var}(X)}{t^2}.$$

## Agenda

- Chernoff Bound
- Example: Server Load, and the union bound

## Chebyshev & Binomial

$\mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq \frac{\mathrm{Var}(X)}{t^2}$.

Reformulated: $P(|X - \mu| \geq \delta\mu) \leq \frac{\sigma^2}{\delta^2\mu^2}$ where $\mu = \mathbb{E}[X]$ and $\sigma^2 = \mathrm{Var}(X)$

If $X \sim \mathrm{Bin}(n, p)$, then $\mu = np$ and $\sigma^2 = np(1-p)$

$$P(|X - \mu| \geq \delta\mu) \leq \frac{np(1-p)}{\delta^2 n^2 p^2} = \frac{1}{\delta^2} \cdot \frac{1}{n} \cdot \frac{1-p}{p}$$

E.g., $\delta = 0.1$, $p = 0.5$: $n = 200$: $P(|X - \mu| \geq \delta\mu) \leq 0.5$

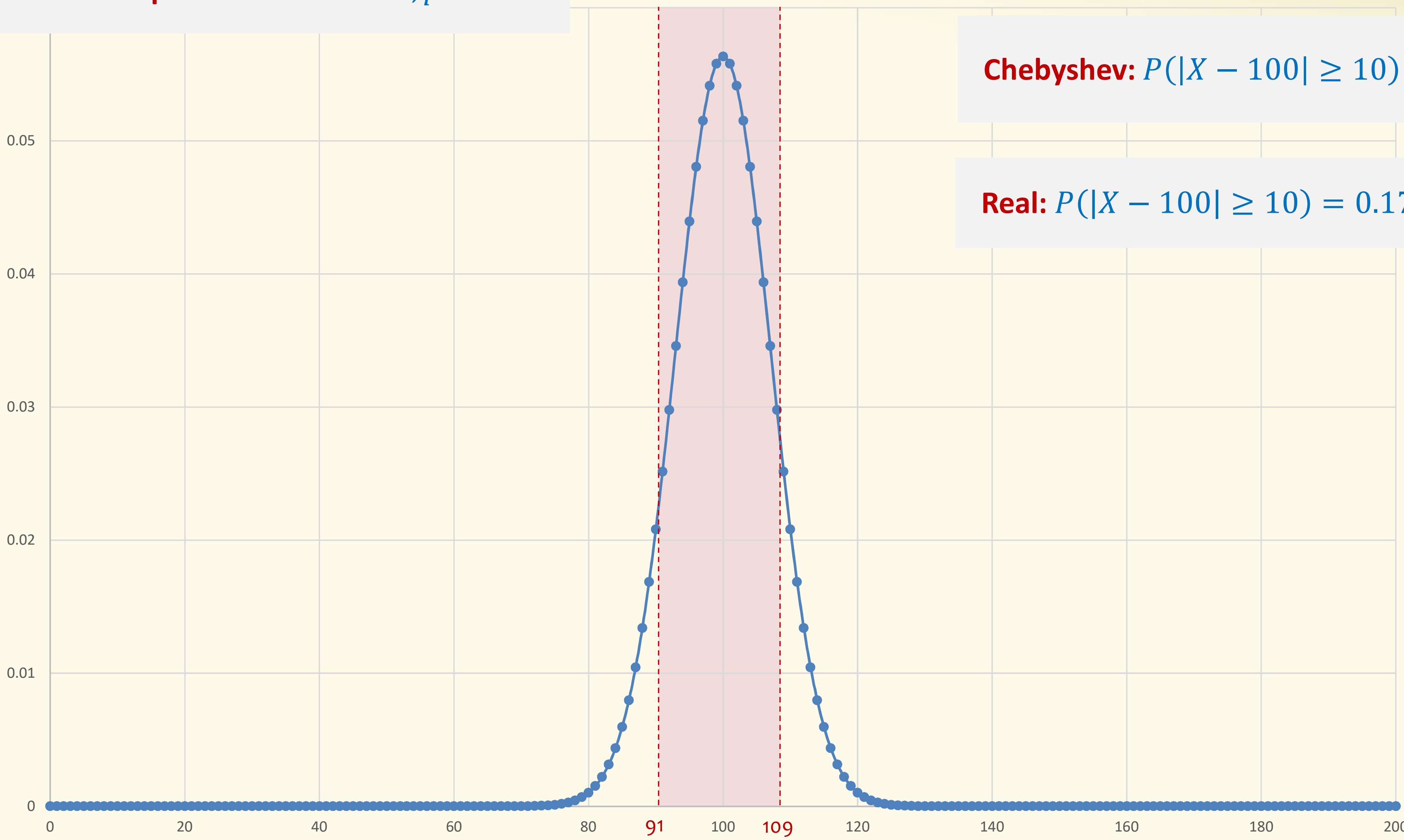
$n = 800$: $P(|X - \mu| \geq \delta\mu) \leq 0.125$

**How good is it?**

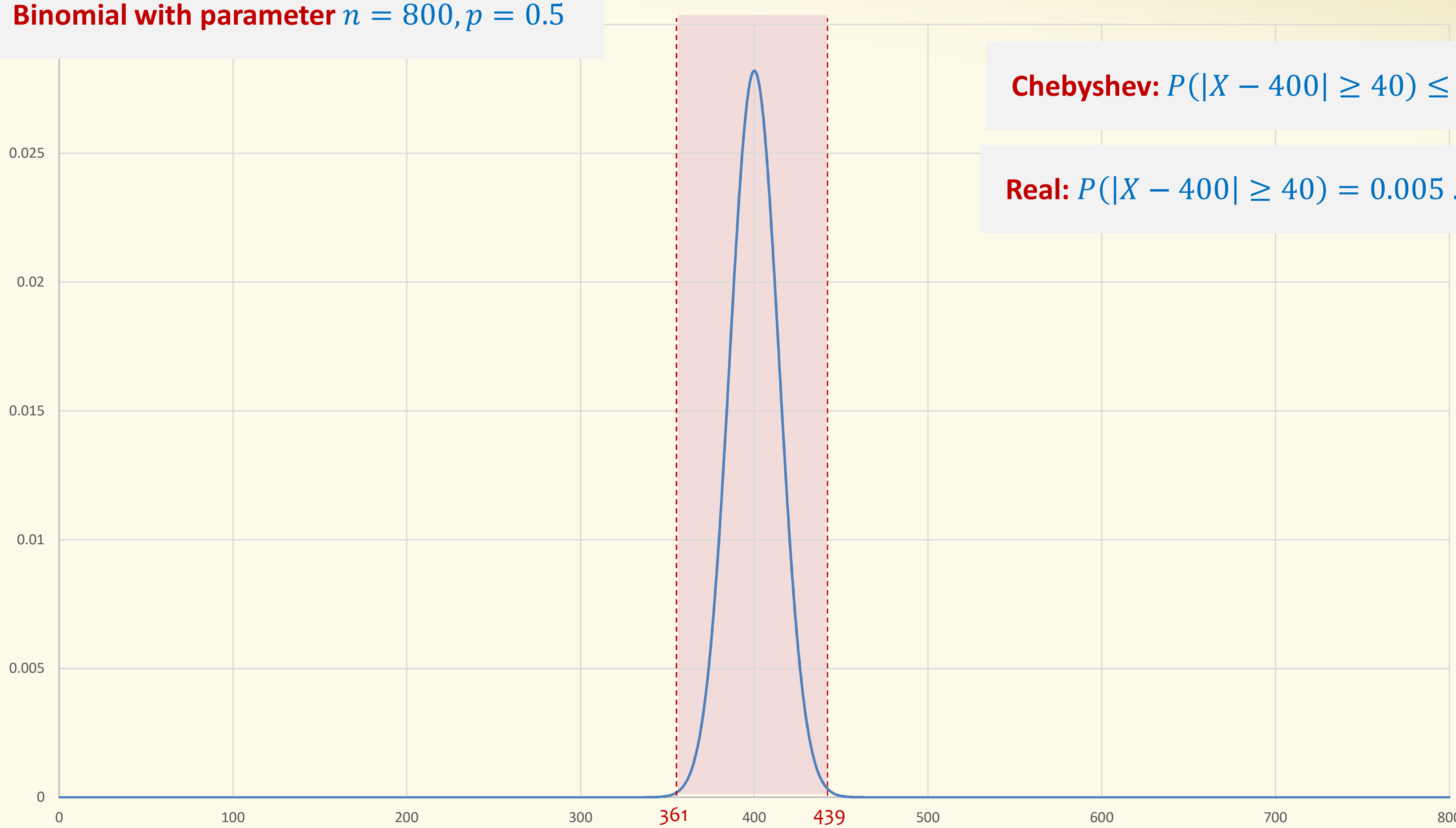
**Binomial with parameter** $n = 200, p = 0.5$

**Chebyshev:** $P(|X - 100| \geq 10) \leq \frac{1}{2}$

**Real:** $P(|X - 100| \geq 10) = 0.179 \ldots$



**Binomial with parameter** $n = 800, p = 0.5$




**Chebyshev:** $P(|X - 400| \geq 40) \leq 0.125$

**Real:** $P(|X - 400| \geq 40) = 0.005 \ldots$

# Chernoff-Hoeffding Bound

**Theorem.** Let $X = X_1 + \cdots + X_n$ be a sum of independent RVs, each taking values in $[0,1]$, such that $\mathbb{E}[X] = \mu$. Then, for every $\delta \in [0,1]$,

$$P(|X - \mu| \geq \delta \cdot \mu) \leq e^{-\frac{\delta^2 \mu}{4}}.$$

Herman Chernoff, Herman Rubin, Wassily Hoeffding

**Example:** If $X \sim \mathrm{Bin}(n, p)$, then $X = X_1 + \cdots + X_n$ is a sum of independent $\{0,1\}$-Bernoulli variables, and $\mu = np$

**Note:** More accurate versions are possible, but with more cumbersome right-hand side (e.g., see textbook)

# Chernoff-Hoeffding Bound – Binomial Distribution

**Theorem. (CH bound, binomial case)** Let $X \sim \mathrm{Bin}(n,p)$. Let $\mu = np = \mathbb{E}[X]$. Then, for any $\delta \in [0,1]$,

$$P(|X-\mu| \geq \delta \cdot \mu) \leq e^{-\frac{\delta^2 np}{4}}.$$

**Example:**

$p = 0.5$

$\delta = 0.1$

| $n$ | Chebyshev $\frac{1}{\delta^2} \cdot \frac{1}{n} \cdot \frac{1-p}{p}$ | Chernoff $e^{-\frac{\delta^2 np}{4}}$ |
|---|---|---|
| 800 | 0.125 | 0.3679 |
| 2600 | 0.03846 | 0.03877 |
| 8000 | 0.0125 | 0.00005 |
| 80000 | 0.00125 | $3.72 \times 10^{-44}$ |

# Chernoff Bound – Example

$$\mathbb{P}(|X - \mu| \geq \delta \cdot \mu) \leq e^{-\frac{\delta^2 np}{4}}.$$

Alice tosses a fair coin $n$ times, what is an upper bound for the probability that she sees heads at least $0.75 \times n$ times?

Poll: pollev.com/paulbeame028

a. $e^{-n/64}$
b. $e^{-n/32}$
c. $e^{-n/16}$
d. $e^{-n/8}$

# Chernoff vs Chebyshev – Summary

$$\frac{1}{\delta^2} \cdot \frac{1}{n} \cdot \frac{1-p}{p}$$

Chebyshev, <u>linear</u> decrease in $n$

vs

Chernoff, <u>exponential</u> decrease in $n$

$$e^{-\frac{\delta^2 np}{4}}$$

# Why is the Chernoff Bound True?

Proof strategy (upper tail): For any $t > 0$:

- $P(X \geq (1+\delta) \cdot \mu) = P\left(e^{tX} \geq e^{t(1+\delta)\cdot\mu}\right)$
- Then, apply Markov + independence:

$$P\left(e^{tX} \geq e^{t(1+\delta)\cdot\mu}\right) \leq \frac{\mathbb{E}[e^{tX}]}{e^{t(1+\delta)\mu}} = \frac{\mathbb{E}[e^{tX_1}] \cdots \mathbb{E}[e^{tX_n}]}{e^{t(1+\delta)\mu}}$$

- Find $t$ minimizing the right-hand-side.

# Brain Break

# Agenda

- Chernoff Bound
- Example: Server Load, and the union bound

# Application – Distributed Load Balancing

We have $k$ processors, and $n \gg k$ jobs.
We want to distribute jobs evenly across processors.

**Strategy:** Each job assigned to a randomly chosen processor!

$X_i$ = load of processor $i$ $X_i \sim \text{Binomial}(n, 1/k)$ $\mathbb{E}[X_i] = n/k$

$X = \max\{X_1, \dots, X_k\}$ = max load of a processor

**Question:** How close is $X$ to $n/k$?

# Distributed Load Balancing

**Claim. (Load of single server)** If $n > 16k \ln k$, then

$$P\left(X_i > \frac{n}{k} + 4\sqrt{\frac{n \ln k}{k}}\right) \leq 1/k^4.$$

**Example:**

- $n = 10^6 \gg k = 1000$
- $\frac{n}{k} + 4\sqrt{n \ln k / k} \approx 1332$
- *"The probability that server* $i$ *processes more than* $1332$ *jobs is at most 1-over-one-trillion!"*

# Distributed Load Balancing

**Claim. (Load of single server)** If $n > 16k \ln k$, then

$$P\left(X_i > \frac{n}{k} + 4\sqrt{\frac{n \ln k}{k}}\right) = P\left(X_i > \frac{n}{k}\left(1 + 4\sqrt{\frac{k \ln k}{n}}\right)\right) \leq 1/k^4.$$

**Proof.** Set $\mu = \mathbb{E}[X_i] = \frac{n}{k}$ and $\delta = 4\sqrt{\frac{k}{n}\ln k} < 4\sqrt{\frac{k}{16k \ln k}\ln k} = 1$

$n > 16k \ln k$

$$P\left(X_i > \mu\left(1 + 4\sqrt{\frac{k \ln k}{n}}\right)\right) = P\big(X_i > \mu(1+\delta)\big)$$

$$\leq P(|X_i - \mu| > \mu\delta)$$

$$\leq e^{-\frac{\delta^2 \mu}{4}} = e^{-4 \ln k} = \frac{1}{k^4}$$

$\delta^2 = 4^2 \cdot \frac{k \ln k}{n}$

so $\delta^2 \mu = 4^2 \ln k$

# What about the maximum load?

**Claim. (Load of single server)** If $n > 16k \ln k$, then

$$P\left(X_i > \frac{n}{k} + 4\sqrt{\frac{n \ln k}{k}}\right) \leq 1/k^4.$$

What about $X = \max\{X_1, \dots, X_k\}$?

Note: $X_1, \dots, X_k$ are <u>not</u> (mutually) independent!

In particular: $X_1 + \cdots + X_k = n$

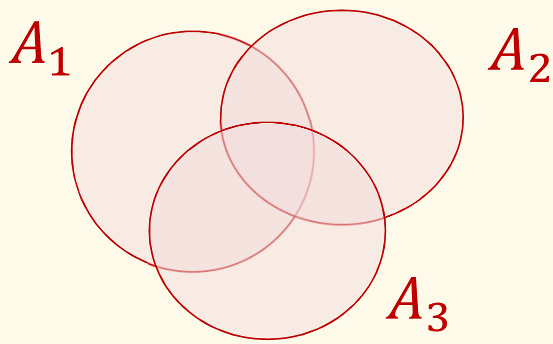
*When non-trivial outcome of one RV can be derived from other RVs, they are non-independent.*

## Detour – Union Bound

**Theorem (Union Bound).** Let $A_1, \dots, A_n$ be arbitrary events. Then,

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

Intuition (3 evts.):



## Detour – Union Bound - Example

Suppose we have $N = 200$ computers, where each one fails with probability $0.001$.

What is the probability that at least one server fails?

Let $A_i$ be the event that server $i$ fails.

Then event that at least one server fails is $\cup_{i=1}^{N} A_i$

$$P\left(\bigcup_{i=1}^{N} A_i\right) \leq \sum_{i=1}^{N} P(A_i) = 0.001N = 0.2$$

# What about the maximum load?

**Claim. (Load of single server)** If $n > 16k \ln k$, then

$$P\left(X_i > \frac{n}{k} + 4\sqrt{\frac{n \ln k}{k}}\right) \leq 1/k^4.$$

What about $X = \max\{X_1, \dots, X_k\}$?

$$P\left(X > \frac{n}{k} + 4\sqrt{n \ln k \,/k}\right) = P\left(\left\{X_1 > \frac{n}{k} + 4\sqrt{n \ln k \,/k}\right\} \cup \cdots \cup \left\{X_k > \frac{n}{k} + 4\sqrt{n \ln k \,/k}\right\}\right)$$

$$\leq P\left(X_1 > \frac{n}{k} + 4\sqrt{\frac{n \ln k}{k}}\right) + \cdots + P\left(X_k > \frac{n}{k} + 4\sqrt{n \ln k \,/k}\right)$$

Union bound

$$\leq \frac{1}{k^4} + \cdots + \frac{1}{k^4} = k \times \frac{1}{k^4} = \frac{1}{k^3}$$

# What about the maximum load?

**Claim. (Load of single server)** If $n > 16k \ln k$, then

$$P\left(X_i > \frac{n}{k} + 4\sqrt{\frac{n \ln k}{k}}\right) \leq 1/k^4.$$

**Claim. (Max load)** Let $X = \max\{X_1, \dots, X_k\}$. If $n > 16k \ln k$, then

$$P\left(X > \frac{n}{k} + 4\sqrt{\frac{n \ln k}{k}}\right) \leq 1/k^3.$$