CSE 312: Foundations of Computing II — Summer 2022

Problem Set 2 (due Friday, July 8, 11:59pm)

Directions: *For each problem, explain/justify how you obtained your answer, as correct answers without an explanation may receive* ***no credit****. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide. Unless you are asked to, you should leave your answer in terms of factorials, combinations, etc., for instance* 26^7 *or* $26!/7!$ *or* $26 \cdot \binom{26}{7}$.

Submission: *You must upload a* ***pdf*** *of your solutions to Gradescope under "Pset 2 [Written]". The use of LaTex is highly recommended, and we have provided a template. (Note that if you want to hand-write your solutions, you'll need to scan them. If we cannot make out your writing, your work may be ungradable, so make sure it is legible.) Your code will be submitted as a .py file under "Pset 2 [Coding]".*

Instructions as to how to upload your solutions to Gradescope are on the course web page.

Remember that you must tag your written problems on Gradescope, or you will potentially receive ***no credit*** *as mentioned in the syllabus. Please put each numbered problem on its own page in the pdf (this will make selecting pages easier when you submit), and ensure that your pdfs are oriented correctly (e.g. not upside-down or sideways). As stated above, the coding problem will also be submitted to Gradescope.*

Collaboration: *This pset must be submitted* ***individually****. You are welcome and encouraged to discuss approaches with your fellow students, but everyone* ***must write up their own solutions.*** *Failure to do so is an instance of academic dishonesty.*

1. Binomial Theorem applications (12 points)

(a) [4 Points] What is the coefficient of x^5y^{12} in the expansion of $(2x - y^2)^{11}$?

(b) [8 Points] Use the binomial theorem to show that

$$\sum_{i=0}^{200} \binom{200}{i} (-4)^{200-i} = 3^{200}$$

2. Combinatorial Proof (16 points)

Prove each of the following identities using a *combinatorial argument*; an algebraic solution will be marked substantially incorrect. (Note that $\binom{a}{b}$ is 0 if $b > a$.)

(a) [8 Points]

$$\sum_{k=0}^{\infty} \binom{m}{k}\binom{n}{k} = \binom{m+n}{n}.$$

Hint: Start with the right hand side and imagine you are choosing a team of n people from a group of people consisting of m Americans and n Canadians.

(b) [8 Points]

$$\sum_{k=0}^{\infty} \binom{n}{k}\binom{k}{m} = \binom{n}{m}2^{n-m}.$$

3. Rotating the table (18 points)

At a dinner party, all of the n people present are to be seated at a circular table. Suppose there is a nametag at each place at the table and suppose that nobody sits down in their correct place. Use the pigeon-hole principle to show that it is possible to rotate the table so that at least two people are sitting in the correct place. Be sure to specify precisely what the pigeons are, precisely what the pigeonholes are, and precisely what the mapping of pigeons to pigeonholes is.

4. Stuff into stuff (14 points)

(a) [4 Points] We have 10 people and 30 rooms. How many different ways are there to assign the (distinguishable) people to the (distinguishable) rooms? (Any number of people can go into any of the 30 rooms.)

(b) [4 Points] We have 20 identical (indistinguishable) apples. How many different ways are there to place the apples into 30 (distinguishable) boxes? (Any number of apples can go into any of the boxes.)

(c) [6 Points] We have 30 identical (indistinguishable) apples. How many different ways are there to place the apples into 8 (distinguishable) boxes, if each box is required to have at least two apples in it.?

5. Random Questions (20 points)

(a) [6 Points] What is the probability that the digit 1 doesn't appear among n digits where each digit is one of (0-9) and all sequences are equally likely?

(b) [6 Points] Suppose you randomly permute the numbers $1, 2, \dots, n$, (where $n > 500$). That is, you select a permutation uniformly at random. What is the probability that the number 3 ends up in the 130-th position in the resulting permutation? (For example, in the permutation $1, 3, 2, 5, 4$ of the numbers $1 \dots 5$, the number 2 is in the 3rd position in the permutation and the number 4 is in the 5th position.)

(c) [8 Points] A fair coin is flipped n times (each outcome in $\{H, T\}^n$ is equally likely). What is the probability that all heads occur at the end of the sequence? (The case that there are no heads is a special case of having all heads at the end of the sequence, i.e. 0 heads.)

6. Ping Pong [coding + written] (20 points)

We'll finally answer the long-awaited question: what's the probability you win a ping pong game up to n points, when your probability of winning each point is p (and your friend wins the point with probability $1-p$)? Assume you have to win by (at least) 2; for example, if $n = 21$ and the score is $21 - 20$, the game isn't over yet.

Write your code for the following parts in the provided file: `cse312_pset2_pingpong.py`.

(a) [5 Points] Implement the function `part_a`.

(b) [15 Points] Implement the function `part_b`. This function will NOT be autograded but you will still submit it; you should use the space here to generate the plot asked of you below.

 (i) Generate a plot similar to the one shown below in Python (without the watermarks). Details on how to construct it are in the starter code. Attach your plot in your written submission for this part. Your plot should:

 - contain plot and axis titles,
 - have the same shape as the plot below,
 - use three different colors,
 - use three different line styles,

- and a legend for the three lines.

(ii) Write AT MOST 2-3 sentences identifying the interesting pattern you notice when n gets larger (regarding the steepness of the curve). Try to explain why it makes sense. (Later in the course, we will see why more formally.)

(iii) Each curve you make for different values of n always (approximately) passes through 3 points. Give the three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, and explain why intuitively this happens in AT MOST 2-3 sentences.

Figure 1: Your plot should look something like this.