CSE 312: Foundations of Computing II

Section 5: Variance, Independence of RVs; Zoo of discrete R.V.s Solutions

1. Review of Main Concepts - Discrete R.V.s

(a) **Variance**: Let X be a random variable and $\mu = \mathbb{E}[X]$. The variance of X is defined to be $\text{Var}(X) = \mathbb{E}\left[(X-\mu)^2\right]$. Notice that since this is an expectation of a nonnegative random variable, i.e., $(X-\mu)^2$, variance is always nonnegative. With some algebra, we can simplify this to $\text{Var}(X) = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$.

(b) **Standard Deviation**: Let X be a random variable. We define the standard deviation of X to be the square root of the variance, and denote it $\sigma = \sqrt{Var(X)}$.

(c) **Property of Variance**: Let $a, b \in \mathbb{R}$ and let X be a random variable. Then, $Var(aX+b) = a^2 Var(X)$.

(d) **Independence**: Random variable X and event E are independent iff

$$\forall x, \quad \mathbb{P}(X = x \cap E) = \mathbb{P}(X = x)\mathbb{P}(E)$$

(e) **Independence**: Random variables X and Y are independent iff

$$\forall x \forall y, \quad \mathbb{P}(X = x \cap Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$$

In this case, we have $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$ (the converse is not necessarily true).

(f) **Independence of functions of a r.v.**: If X and Y are independent and $g(\cdot), h(\cdot)$ are functions mapping real numbers to real numbers, then $g(X)$ and $h(Y)$ are independent. (See if you can prove this!)

(g) **i.i.d. (independent and identically distributed)**: Random variables $X_1, \ldots, X_n$ are i.i.d. (or iid) iff they are independent and have the same probability mass function.

(h) **Variance of Independent Variables**: If X is independent of Y, $\text{Var}\,(X+Y) = \text{Var}\,(X) + \text{Var}(Y)$. This depends on independence, whereas linearity of expectation always holds. Note that this combined with the above shows that $\forall a, b, c \in \mathbb{R}$ and if X is independent of Y, $\text{Var}(aX + bY + c) = a^2\text{Var}(X) + b^2\text{Var}(Y)$.

2. Review: Zoo of Discrete Random Variables

(a) **Uniform**: $X \sim \text{Uniform}(a, b)$ ($\text{Unif}(a, b)$ for short), for integers $a \le b$, iff X has the following probability mass function:

$$p_X(k) = \frac{1}{b-a+1}, \quad k = a, a+1, \ldots, b$$

$\mathbb{E}[X] = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a)(b-a+2)}{12}$. This represents each integer from $[a, b]$ to be equally likely. For example, a single roll of a fair die is $\text{Uniform}(1, 6)$.

(b) **Bernoulli (or indicator)**: $X \sim \text{Bernoulli}(p)$ ($\text{Ber}(p)$ for short) iff X has the following probability mass function:

$$p_X(k) = \begin{cases} p, & k = 1 \\ 1-p, & k = 0 \end{cases}$$

$\mathbb{E}[X] = p$ and $Var(X) = p(1-p)$. An example of a Bernoulli r.v. is one flip of a coin with $\mathbb{P}(\text{head}) = p$.

(c) **Binomial**: $X \sim \text{Binomial}(n, p)$ ($\text{Bin}(n, p)$ for short) iff X is the sum of n iid $\text{Bernoulli}(p)$ random variables. X has probability mass function

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

$\mathbb{E}[X] = np$ and $Var(X) = np(1-p)$. An example of a Binomial r.v. is the number of heads in n independent flips of a coin with $\mathbb{P}(\text{head}) = p$. Note that $\text{Bin}(1, p) \equiv \text{Ber}(p)$. As $n \to \infty$ and $p \to 0$, with $np = \lambda$, then $\text{Bin}\,(n, p) \to \text{Poi}(\lambda)$. If $X_1, \ldots, X_n$ are independent Binomial r.v.'s, where $X_i \sim \text{Bin}(N_i, p)$, then $X = X_1 + \ldots + X_n \sim \text{Bin}(N_1 + \ldots + N_n, p)$.

(d) **Geometric:** $X \sim$ Geometric(p) (Geo(p) for short) iff X has the following probability mass function:

$$p_X(k) = (1-p)^{k-1} p, \quad k = 1, 2, \ldots$$

$\mathbb{E}[X] = \frac{1}{p}$ and $Var(X) = \frac{1-p}{p^2}$. An example of a Geometric r.v. is the number of independent coin flips up to and including the first head, where $\mathbb{P}(\text{head}) = p$.

(e) **Poisson**: $X \sim$ Poisson(λ) (Poi(λ) for short) iff X has the following probability mass function:

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots$$

$\mathbb{E}[X] = \lambda$ and $Var(X) = \lambda$. An example of a Poisson r.v. is the number of people born during a particular minute, where λ is the average birth rate per minute. If $X_1, \ldots, X_n$ are independent Poisson r.v.'s, where $X_i \sim$ Poi(λ_i), then $X = X_1 + \ldots + X_n \sim$ Poi$(\lambda_1 + \ldots + \lambda_n)$.

(f) **Negative Binomial** : $X \sim$ NegativeBinomial(r, p) (NegBin(r, p) for short) iff X is the sum of r iid Geometric(p) random variables. X has probability mass function

$$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \ldots$$

$\mathbb{E}[X] = \frac{r}{p}$ and $Var(X) = \frac{r(1-p)}{p^2}$. An example of a Negative Binomial r.v. is the number of independent coin flips up to and including the r^{th} head, where $\mathbb{P}(\text{head}) = p$. If $X_1, \ldots, X_n$ are independent Negative Binomial r.v.'s, where $X_i \sim$ NegBin(r_i, p), then $X = X_1 + \ldots + X_n \sim$ NegBin$(r_1 + \ldots + r_n, p)$.

(g) **Hypergeometric** : $X \sim$ HyperGeometric(N, K, n) (HypGeo(N, K, n) for short) iff X has the following probability mass function:

$$p_X(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}, \quad k = \max\{0, n+K-N\}, \ldots, \min\{K, n\}$$

$\mathbb{E}[X] = n\frac{K}{N}$. This represents the number of successes drawn, when n items are drawn from a bag with N items (K of which are successes, and $N-K$ failures) without replacement. If we did this with replacement, then this scenario would be represented as Bin $\left(n, \frac{K}{N}\right)$.

3. Review of Main Concepts - Continuous R.V.s

(a) **Cumulative Distribution Function (cdf)**: For any random variable (discrete or continuous) X, the cumulative distribution function is defined as $F_X(x) = \mathbb{P}(X \le x)$. Notice that this function must be monotonically nondecreasing: if $x < y$ then $F_X(x) \le F_X(y)$, because $\mathbb{P}(X \le x) \le \mathbb{P}(X \le y)$. Also notice that since probabilities are between 0 and 1, that $0 \le F_X(x) \le 1$ for all x, with $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to +\infty} F_X(x) = 1$.

(b) **Continuous Random Variable**: A continuous random variable X is one for which its cumulative distribution function $F_X(x) :\to$ is continuous everywhere. A continuous random variable has an uncountably infinite number of values.

(c) **Probability Density Function (pdf or density)**: Let X be a continuous random variable. Then the probability density function $f_X(x) :\to$ of X is defined as $f_X(x) = \frac{d}{dx} F_X(x)$. Turning this around, it means that $F_X(x) = \mathbb{P}(X \le x) = \int_{-\infty}^{x} f_X(t)\, dt$. From this, it follows that $\mathbb{P}(a \le X \le b) = F_X(b) - F_X(a) = \int_a^b f_X(x) dx$ and that $\int_{-\infty}^{\infty} f_X(x) dx = 1$. From the fact that $F_X(x)$ is monotonically nondecreasing it follows that $f_X(x) \ge 0$ for every real number x.

If X is a continuous random variable, note that in general $f_X(a) \ne \mathbb{P}(X = a)$, since $\mathbb{P}(X = a) = F_X(a) - F_X(a) = 0$ for all a. However, the probability that X is close to a is proportional to $f_X(a)$: for small δ, $\mathbb{P}\left(a - \frac{\delta}{2} < X < a + \frac{\delta}{2}\right) \approx \delta f_X(a)$.

(d) **i.i.d. (independent and identically distributed)**: Random variables $X_1, \ldots, X_n$ are i.i.d. (or iid) if they are independent and have the same probability mass function or probability density function.

(e) **Discrete to Continuous:**

	Discrete	**Continuous**
PMF/PDF	$p_X(x) = \mathbb{P}(X = x)$	$f_X(x) \neq \mathbb{P}(X = x) = 0$
CDF	$F_X(x) = \sum_{t \le x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
Expectation	$\mathbb{E}[X] = \sum_x x p_X(x)$	$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$
LOTUS	$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

(f) **Standardizing**: Let X be any random variable (discrete or continuous, not necessarily normal), with $\mathbb{E}[X] = \mu$ and $Var(X) = \sigma^2$. If we let $Y = \frac{X-\mu}{\sigma}$, then $\mathbb{E}[Y] = 0$ and $Var(Y) = 1$.

4. Pond Fishing

Suppose I am fishing in a pond with B blue fish, R red fish, and G green fish, where $B + R + G = N$. For each of the following scenarios, identify the most appropriate distribution (with parameter(s)):

(a) how many of the next 10 fish I catch are blue, if I catch and release

Solution:

Since this is the same as saying how many of my next 10 trials (fish) are a success (are blue), this is a binomial distribution. Specifically, since we are doing catch and release, the probability of a given fish being blue is $\frac{B}{N}$ and each trial is independent. Thus:

$$\text{Bin}\left(10, \frac{B}{N}\right)$$

(b) how many fish I had to catch until my first green fish, if I catch and release

Solution:

Once again, each catch is independent, so this is asking how many trials until we see a success, hence it is a geometric distribution:

$$\text{Geo}\left(\frac{G}{N}\right)$$

(c) how many red fish I catch in the next five minutes, if I catch on average r red fish per minute

Solution:

This is asking for the number of occurrences of event given an average rate, which is the definition of the Poisson distribution. Since we're looking for events in the next 5 minutes, that is our time unit, so we have to adjust the average rate to match (r per minute becomes $5r$ per 5 minutes).

$$\text{Poi}(5r)$$

(d) whether or not my next fish is blue

Solution:

This is the same as the binomial case, but it's only one trial, so it is necessarily Bernoulli.

$$\text{Ber}\left(\frac{B}{N}\right)$$

(e) how many of the next 10 fish I catch are blue, if I do not release the fish back to the pond after each catch

Solution:

This is a hypergeometric r.v. Its definition is the number of successes in n draws (without replacement) from N items that contain K successes in total. In this case, we have 10 draws (without replacement because we do not catch and release), and out of the N fish, B are blue (a success).

$$\text{HypGeo}(N, B, 10)$$

(f) how many fish I have to catch until I catch three red fish, if I catch and release

Solution:

This is a negative binomial r.v. It models the number of trials with probability of success p, until you get r successes. In this case, as before, our trials are caught fish (with replacement this time) and our success is if the fish are red, which happens with probability $\frac{R}{N}$.

$$\text{NegBin}\left(3, \frac{R}{N}\right)$$

5. Best Coach Ever!!

You are a hardworking boxer. Your coach tells you that the probability of your winning a boxing match is 0.2 independently of every other match.

(a) How many matches do you expect to fight until you win 10 times and what kind of random variable is this?

Solution:

The number of matches you have to fight until you win 10 times can be modeled by $\sum_{i=1}^{10} X_i$ where $X_i \sim \text{Geometric}(0.2)$ is the number of matches you have to fight to go from $i-1$ wins to i wins, including the match that gets you your i^{th} win, where every match has a 0.2 probability of success. Recall $\mathbb{E}[X_i] = \frac{1}{0.2} = 5$. $\mathbb{E}\left[\sum_{i=1}^{10} X_i\right] = \sum_{i=1}^{10} \mathbb{E}[X_i] = \sum_{i}^{10} \frac{1}{0.2} = 10 \cdot 5 = 50$.

Note that since $X = \sum_{i=1}^{10} X_i$ is the sum of iid geometric random variables, we instead could've said that $X \sim \text{NegBin}(10, 0.2)$ and found that $\mathbb{E}[X] = \frac{10}{0.2} = 50$, the same as above.

(b) You only get to play 12 matches every year. To win a spot in the Annual Boxing Championship, a boxer needs to win at least 10 matches in a year. What is the probability that you will go to the Championship this year and what kind of random variable is the number of matches you win out of the 12?

Solution:

You can go to the championship if you win more than or equal to 10 times this year. Let Y be the number of matches you win out of the 12 matches. Note that $Y \sim \text{Binomial}(12, 0.2)$. Since the max number you can win is 12 (there are 12 matches), we are looking for $P(10 \le Y \le 12)$. Thus, since Y is discrete, we are interested in

$$\mathbb{P}(Y=10) + \mathbb{P}(Y=11) + \mathbb{P}(Y=12) = \sum_{i=10}^{12} \binom{12}{i} 0.2^i (1-0.2)^{12-i} \approx 0.0000045$$

(c) Let p be your answer to part (b). How many times can you expect to go to the Championship in your 20 year career?

Solution:

The number of times you go to the championship can be modeled by $Y \sim$ Binomial$(20, p)$. So, $E[Y] = 20 \cdot p$.

6. Variance of a Product

Let X, Y, Z be independent random variables with means μ_X, μ_Y, μ_Z and variances $\sigma_X^2, \sigma_Y^2, \sigma_Z^2$, respectively. Find $Var(XY - Z)$.

Solution:

First notice that $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \implies \mathbb{E}[X^2] = Var(X) + \mathbb{E}[X]^2 = \sigma_X^2 + \mu_X^2$, and same for Y.

$$\begin{aligned} Var(XY) &= \mathbb{E}[X^2Y^2] - \mathbb{E}[XY]^2 \text{ (by theorem in class)} \\ &= \mathbb{E}[X^2]\,\mathbb{E}[Y^2] - (\mathbb{E}[X]\,\mathbb{E}[Y])^2 \text{(by independence)} \\ &= \mathbb{E}[X^2]\,\mathbb{E}[Y^2] - \mathbb{E}[X]^2\,\mathbb{E}[Y]^2 \\ &= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2\mu_Y^2 \end{aligned}$$

By independence,

$$\begin{aligned} Var(XY - Z) &= Var(XY) + Var(-Z) = Var(XY) + Var(Z) \\ &= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2\mu_Y^2 + \sigma_Z^2 \end{aligned}$$

7. True or False?

Identify the following statements as true or false (true means always true). Justify your answer.

(a) For any random variable X, we have $\mathbb{E}[X^2] \geq \mathbb{E}[X]^2$.

Solution:

True, since $0 \leq Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$, since the squaring necessitates the result is non-negative.

(b) Let X, Y be random variables. Then, X and Y are independent if and only if $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$.

Solution:

False. The forward implication is true, but the reverse is not. For example, if $X \sim$ Uniform$(-1, 1)$ (equally likely to be in $\{-1, 0, 1\}$), and $Y = X^2$, we have $\mathbb{E}[X] = 0$, so $\mathbb{E}[X]\,\mathbb{E}[Y] = 0$. However, since $X = X^3$ (why?), $\mathbb{E}[XY] = \mathbb{E}[XX^2] = \mathbb{E}[X^3] = \mathbb{E}[X] = 0$, we have that $\mathbb{E}[X]\,\mathbb{E}[Y] = 0 = \mathbb{E}[XY]$. However, X and Y are not independent; indeed, $\mathbb{P}(Y = 0|X = 0) = 1 \neq \frac{1}{3} = \mathbb{P}(Y = 0)$.

(c) Let $X \sim$ Binomial(n, p) and $Y \sim$ Binomial(m, p) be independent. Then, $X + Y \sim$ Binomial$(n + m, p)$.

Solution:

True. X is the sum of n independent Bernoulli trials, and Y is the sum of m. So $X + Y$ is the sum of $n + m$ independent Bernoulli trials, so $X + Y \sim$ Binomial$(n + m, p)$.

(d) Let $X_1, ..., X_{n+1}$ be independent Bernoulli(p) random variables. Then, $\mathbb{E}[\sum_{i=1}^{n} X_iX_{i+1}] = np^2$.

Solution:

True. Notice that X_iX_{i+1} is also Bernoulli (only takes on 0 and 1), but is 1 iff both are 1, so $X_iX_{i+1} \sim$ Bernoulli(p^2). The statement holds by linearity, since $\mathbb{E}[X_iX_{i+1}] = p^2$.

(e) Let $X_1, ..., X_{n+1}$ be independent Bernoulli(p) random variables. Then, $Y = \sum_{i=1}^{n} X_iX_{i+1} \sim$ Binomial(n, p^2).

Solution:

False. They are all Bernoulli p^2 as determined in the previous part, but they are not independent. Indeed, $\mathbb{P}(X_1X_2 = 1|X_2X_3 = 1) = \mathbb{P}(X_1 = 1) = p \neq p^2 = \mathbb{P}(X_1X_2 = 1)$.

(f) If $X \sim$ Bernoulli(p), then $nX \sim$ Binomial(n, p).

Solution:

False. The range of X is $\{0, 1\}$, so the range of nX is $\{0, n\}$. nX cannot be Bin(n, p), otherwise its range would be $\{0, 1, ..., n\}$.

(g) If $X \sim$ Binomial(n, p), then $\frac{X}{n} \sim$ Bernoulli(p).

Solution:

False. Again, the range of X is $\{0, 1, ..., n\}$, so the range of $\frac{X}{n}$ is $\{0, \frac{1}{n}, \frac{2}{n}, ..., 1\}$. Hence it cannot be Ber(p), otherwise its range would be $\{0, 1\}$.

(h) For any two independent random variables X, Y, we have $Var(X - Y) = Var(X) - Var(Y)$.

Solution:

False. $Var(X - Y) = Var(X + (-Y)) = Var(X) + (-1)^2Var(Y) = Var(X) + Var(Y)$.

8. Fun with Poissons

Let $X \sim Poisson(\lambda_1)$ and $Y \sim Poisson(\lambda_2)$, and X and Y are independent.

(a) Show that $X + Y \sim Poisson(\lambda_1 + \lambda_2)$

Solution:

To show that a random variable is distributed according to a particular distribution, we must show that they have the same PMF. Thus, we are trying to show that $P(X+Y=n) = e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}$

$$\begin{aligned}
P(X+Y=n) &= \sum_{k=0}^{n} P(X=k \cap Y=n-k) \\
&= \sum_{k=0}^{n} P(X=k)P(Y=n-k) && \text{[X and Y are independent]} \\
&= \sum_{k=0}^{n} e^{-\lambda_1}\frac{\lambda_1^k}{k!}\, e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!} \\
&= e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \frac{\lambda_1^k}{k!}\, \frac{\lambda_2^{n-k}}{(n-k)!} \\
&= e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \frac{1}{k!(n-k)!}\, \lambda_1^k \lambda_2^{n-k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\, \lambda_1^k \lambda_2^{n-k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \binom{n}{k}\, \lambda_1^k \lambda_2^{n-k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\, (\lambda_1+\lambda_2)^n && \text{[Binomial Theorem]}
\end{aligned}$$

(b) Show that $P(X=k \mid X+Y=n) = P(W=k)$ where $W \sim Bin(n, \frac{\lambda_1}{\lambda_1+\lambda_2})$

Solution:

$$\begin{aligned}
P(X=k \mid X+Y=n) &= \frac{P(X=k \cap X+Y=n)}{P(X+Y=n)} \\
&= \frac{P(X=k \cap Y=n-k)}{P(X+Y=n)} \\
&= \frac{P(X=k)P(Y=n-k)}{P(X+Y=n)} && \text{[X and Y are independent]} \\
&= \frac{e^{-\lambda_1}\frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!}}{e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}} \\
&= \frac{\frac{\lambda_1^k}{k!} \cdot \frac{\lambda_2^{n-k}}{(n-k)!}}{\frac{(\lambda_1+\lambda_2)^n}{n!}} \\
&= \frac{n!}{k!(n-k)!} \cdot \frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} \\
&= \binom{n}{k} \frac{\lambda_1^k \, \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^k \, (\lambda_1+\lambda_2)^{n-k}} \\
&= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k} \\
&= \binom{n}{k} p^k \, (1-p)^{n-k} \text{ , where } p = \frac{\lambda_1}{\lambda_1+\lambda_2}
\end{aligned}$$

9. Memorylessness

We say that a random variable X is memoryless if $\mathbb{P}(X > k+i \mid X > k) = \mathbb{P}(X > i)$ for all non-negative integers k and i. The idea is that X does not *remember* its history. Let $X \sim Geo(p)$. Show that X is memoryless.

Solution:

Let's note that if $X \sim Geo(p)$, then $\mathbb{P}(X > k) = \mathbb{P}(\text{no successes in the first } k \text{ trials}) = (1-p)^k$.

$$\begin{aligned}
\mathbb{P}(X > k+i \mid X > k) &= \frac{\mathbb{P}(X > k \mid X > k+i)\, \mathbb{P}(X > k+i)}{\mathbb{P}(X > k)} && \text{[Bayes Theorem]} \\
&= \frac{\mathbb{P}(X > k+i)}{\mathbb{P}(X > k)} && [\mathbb{P}(X > k \mid X > k+i) = 1] \\
&= \frac{(1-p)^{k+i}}{(1-p)^k} && [\mathbb{P}(X > k) = (1-p)^k] \\
&= (1-p)^i \\
&= \mathbb{P}(X > i)
\end{aligned}$$

10. New PDF?

Aleks came up with a function that he thinks could represent a probability density function. He defined the potential pdf for X as $f(x) = \frac{1}{1+x^2}$ defined on $[0, \infty)$. Is this a valid pdf? If not, find a constant c such that the pdf $f_X(x) = \frac{c}{1+x^2}$ is valid. Then find $\mathbb{E}[X]$. (Hints: $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, $\tan \frac{\pi}{2} = \infty$, and $\tan 0 = 0$.)

Solution:
The area under the PDF is 1. So,

$$\int_0^\infty \frac{c}{1+x^2}dx = c\tan^{-1} x \,|_0^\infty = c\left(\frac{\pi}{2} - 0\right) = 1$$

Solving for c gives us $c = 2/\pi$. Using our value we found for c, and the definition of expectation we can compute $E[X]$ as follows:

$$\mathbb{E}[X] = \int_0^\infty \frac{cx}{1+x^2}dx = \frac{2}{\pi}\int_0^\infty \frac{x}{1+x^2}dx = \frac{1}{\pi}\ln(1+x^2) \,|_0^\infty = \infty$$

11. Throwing a dart

Consider the closed unit circle of radius r, i.e., $S = \{(x,y) : x^2 + y^2 \le r^2\}$. Suppose we throw a dart onto this circle and are guaranteed to hit it, but the dart is equally likely to land anywhere in S. Concretely this means that the probability that the dart lands in any particular area of size A (that is entirely inside the circle of radius R), is equal to $\frac{\text{A}}{\text{Area of whole circle}}$. The density outside the circle of radius r is 0.
Let X be the distance the dart lands from the center. What is the CDF and pdf of X? What is $\mathbb{E}[X]$ and $Var(X)$?

Solution:
Since $F_X(x)$ is the probability that the dart lands inside the circle of radius x, that probability is the area of a circle of radius x divided by the area of the circle of radius r (i.e., $\pi x^2/\pi r^2$). Thus, our CDF looks like

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{r^2} & 0 < x \le r \\ 1 & x > r \end{cases}$$

To find the PDF we just need to take the derivative of the CDF, which give us the following:

$$f_X(x) = \begin{cases} \frac{2x}{r^2} & 0 < x \le r \\ 0 & \text{otherwise} \end{cases}$$

Using the definition of expectation we get

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^r x\frac{2x}{r^2}dx = \frac{2}{3r^2}\left(x^3\Big|_0^r\right) = \frac{2}{3}r$$

We know that $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_0^r x^2\frac{2x}{r^2}dx = \frac{2}{4r^2}\left(x^4\Big|_0^r\right) = \frac{1}{2}r^2$$

Plugging this into our variance equation gives

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{1}{2}r^2 - \left(\frac{2}{3}r\right)^2 = \frac{1}{18}r^2$$

12. A square dartboard?

You throw a dart at an $s \times s$ square dartboard. The goal of this game is to get the dart to land as close to the lower left corner of the dartboard as possible. However, your aim is such that the dart is equally likely to land at any point on the dartboard. Let random variable X be the length of the side of the smallest *square* B in the lower left corner of the dartboard that contains the point where the dart lands. That is, the lower left corner of

B must be the same point as the lower left corner of the dartboard, and the dart lands somewhere along the upper or right edge of B. For X, find the CDF, PDF, $\mathbb{E}[X]$, and $Var(X)$.

Solution:

Since $F_X(x)$ is the probability that the dart lands inside the square of side length x, that probability is the area of a square of length x divided by the area of the square of length radius s (i.e., x^2/r^2). Thus, our CDF looks like

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ x^2/s^2, & \text{if } 0 \le x \le s \\ 1, & \text{if } x > s \end{cases}$$

To find the PDF, we just need to take the derivative of the CDF, which gives us the following:

$$f_X(x) = \frac{d}{dx}F_X(x) = \begin{cases} 2x/s^2, & \text{if } 0 \le x \le s \\ 0, & \text{otherwise} \end{cases}$$

Using the definition of expectation and variance we can compute $\mathbb{E}[X]$ and $Var(X)$ in the following manner:

$$\mathbb{E}[X] = \int_0^s x f_X(x) dx = \int_0^s \frac{2x^2}{s^2} dx = \frac{2}{s^2}\int_0^s x^2 dx = \frac{2}{3s^2}\left[x^3\right]_0^s = \frac{2}{3}s$$

$$\mathbb{E}\left[X^2\right] = \int_0^s x^2 f_X(x) dx = \int_0^s \frac{2x^3}{s^2} dx = \frac{2}{s^2}\int_0^s x^3 dx = \frac{1}{2s^2}\left[x^4\right]_0^s = \frac{1}{2}s^2$$

$$\mathsf{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2 = \frac{1}{2}s^2 - \left(\frac{2}{3}s\right)^2 = \frac{1}{18}s^2$$