

CSE 312

Foundations of Computing II

Lecture 3: Counting III



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Slide Credit: Based on Stefano Tessaro's slides for 312 19au incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & myself 😊

Agenda

- Recap
- Inclusion-Exclusion
- Binomial Theorem
- Pigeonhole Principle
- Combinatorial Proofs



$$\binom{n}{k}$$

Recap

- Permutations

$$P(n, k) = \frac{n!}{(n - k)!}$$

- Combinations

$$\binom{n}{k} = \frac{n!}{(n - k)! k!}$$


Recap

- Multinomial Coefficients (bonus content in textbook, 1.2)
- Stars and Bars

Ways to distribute n indistinguishable balls into k distinct bins =

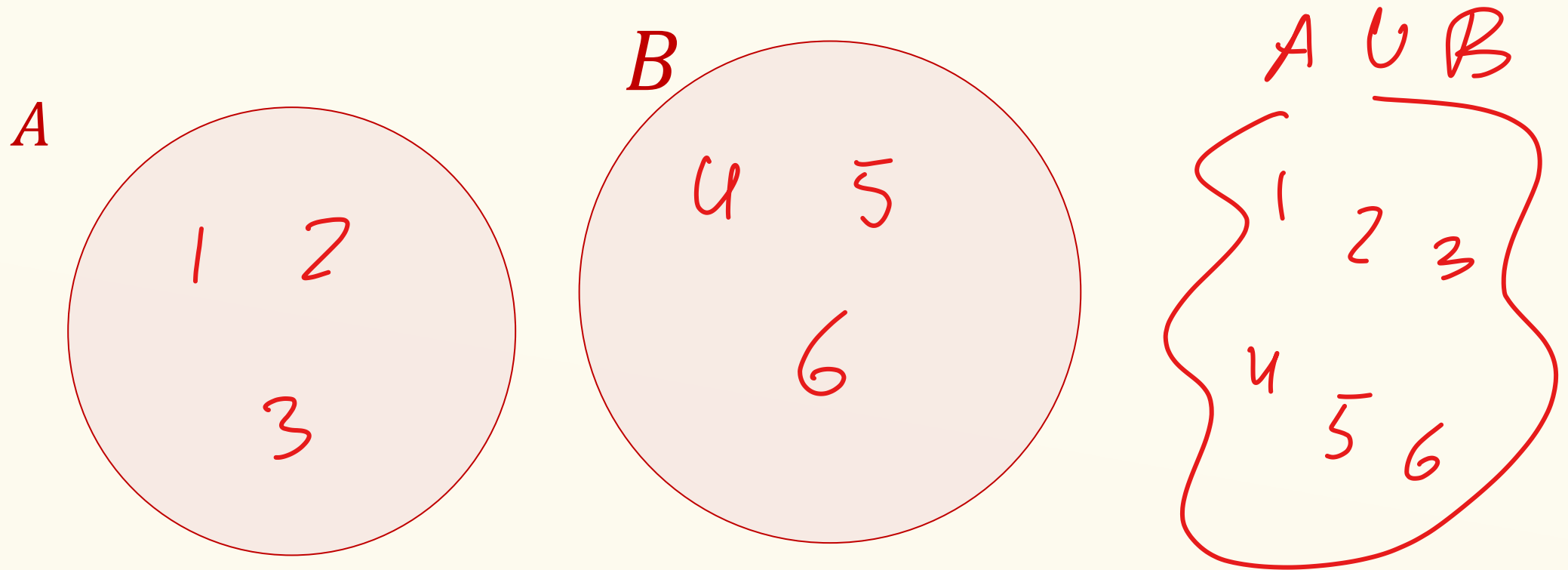
$$\binom{n+k-1}{k-1}$$

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Recap Disjoint Sets

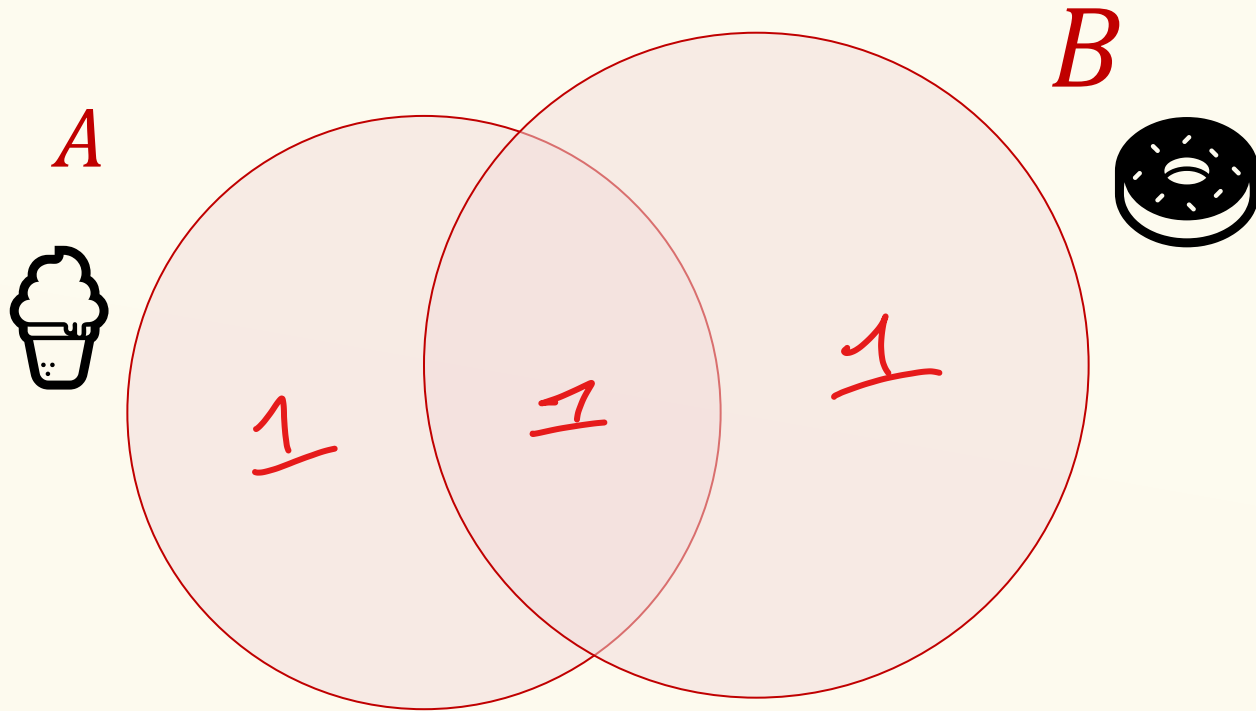
Sets that do not contain common elements ($A \cap B = \emptyset$)



Sum Rule: $|A \cup B| = |A| + |B|$

Inclusion-Exclusion

But what if the sets are not disjoint?

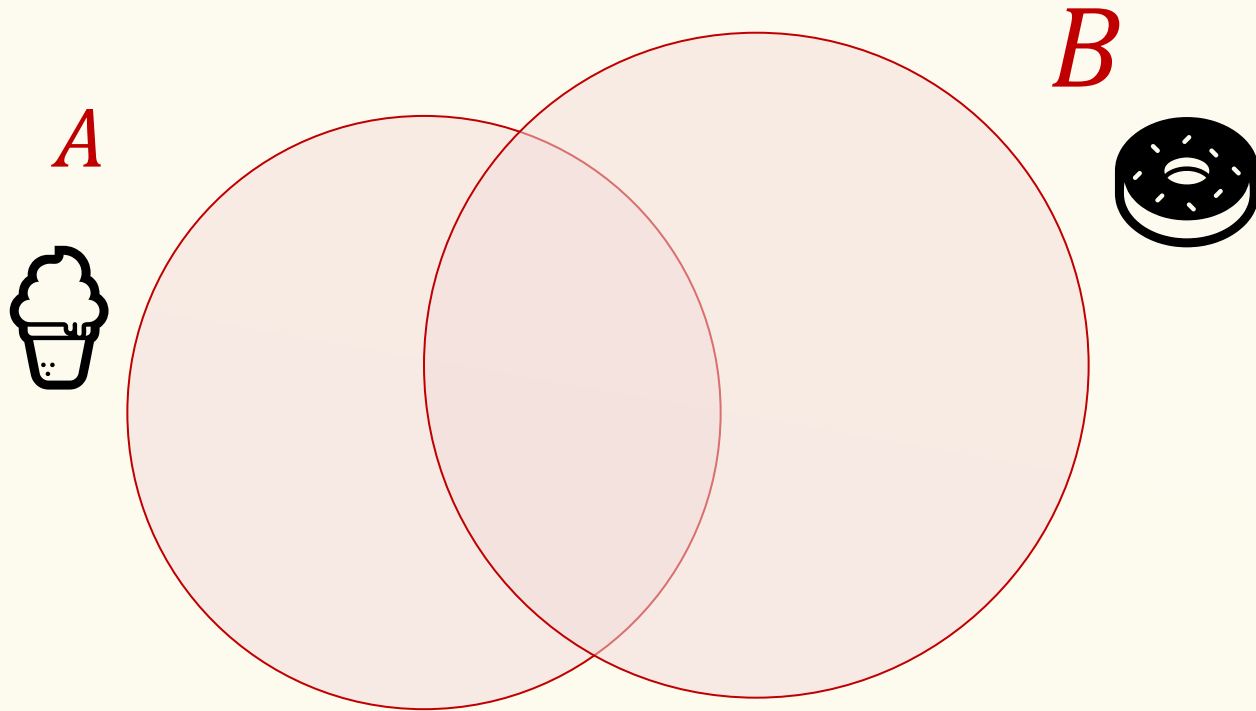


$$\begin{aligned} |A| &= 43 \\ |B| &= 20 \\ |A \cap B| &= 7 \\ |A \cup B| &= ??? \end{aligned}$$

$$43 + 20 - 7$$

Inclusion-Exclusion

But what if the sets are not disjoint?



$$|A| = 43$$

$$|B| = 20$$

$$|A \cap B| = 7$$

$$|A \cup B| = ???$$

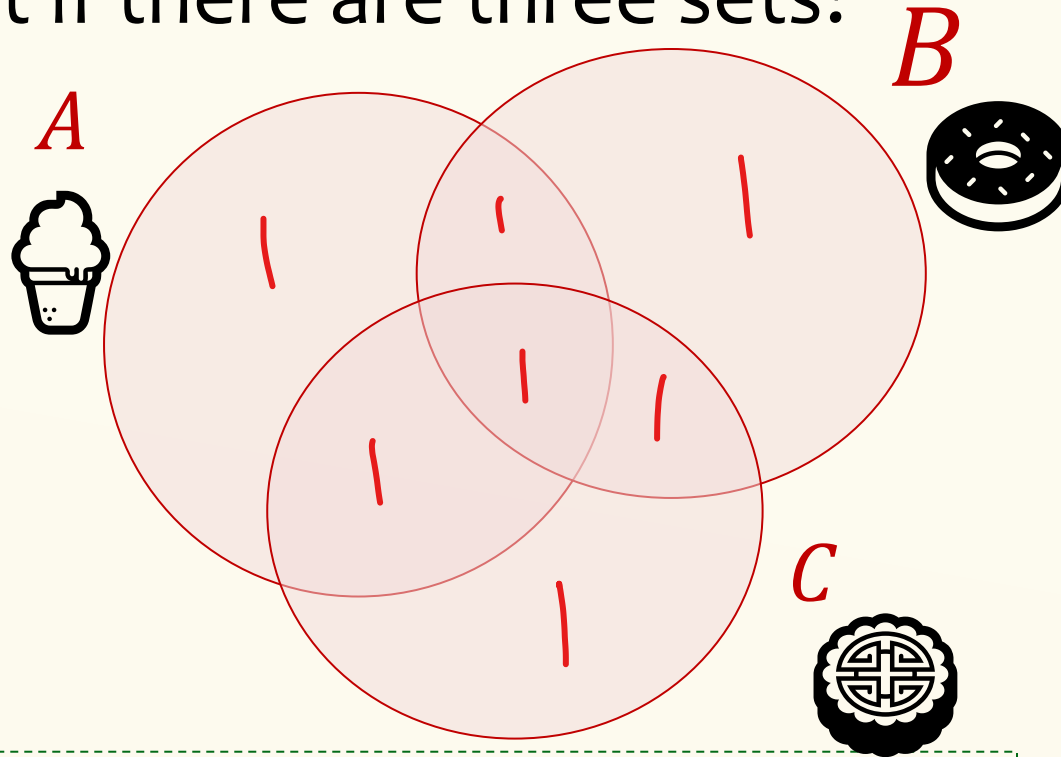
$$|A \cap B| = 0$$

Fact. $|A \cup B| = |A| + |B| - |A \cap B|$

Inclusion-Exclusion

Not drawn to scale

What if there are three sets?



$$|A| = 43$$

$$|B| = 20$$

$$|C| = 35$$

$$|A \cap B| = 7$$

$$|A \cap C| = 16$$

$$|B \cap C| = 11$$

$$|A \cap B \cap C| = 4$$

$$|A \cup B \cup C| = ???$$

Fact.

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - |A \cap B| - |A \cap C| - |B \cap C| \\ &\quad + |A \cap B \cap C| \end{aligned}$$

Inclusion-Exclusion


Let A, B be sets. Then

$$\underline{|A \cup B| = |A| + |B| - |A \cap B|}$$

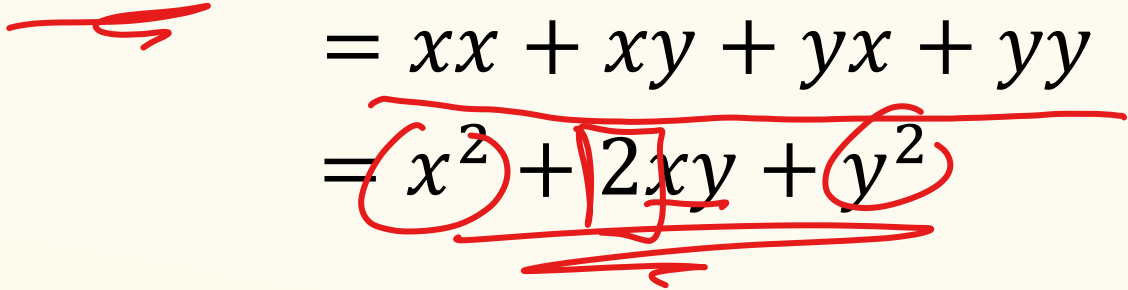
In general, if A_1, A_2, \dots, A_n are sets, then

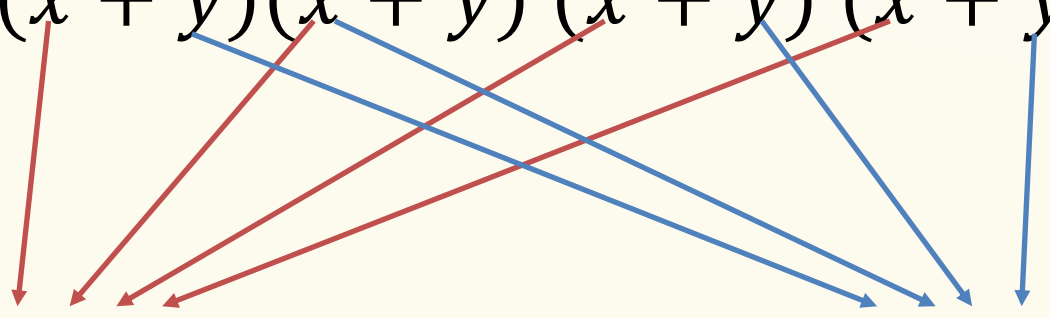
$$\begin{aligned} |A_1 \cup A_2 \cup \dots \cup A_n| &= \textit{singles} - \textit{doubles} + \textit{triples} - \textit{quads} + \dots \\ &= (|A_1| + \dots + |A_n|) - (|A_1 \cap A_2| + \dots + |A_{n-1} \cap A_n|) + \dots \end{aligned}$$

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- Recap
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- **Binomial Theorem** 
- Pigeonhole Principle
- Combinatorial Proofs

Binomial Theorem: Idea

$$\begin{aligned}(x + y)^2 &= (x + y)(x + y) \\ &= xx + xy + yx + yy \\ &= x^2 + 2xy + y^2\end{aligned}$$


$$\begin{aligned}(x + y)^4 &= (x + y)(x + y)(x + y)(x + y) \\ &= xxxx + yyyy + xyxy + yxyy + \dots\end{aligned}$$


Binomial Theorem: Idea

Poll: What is the coefficient for xy^3 ?
3

- A. 4
- B. $\binom{4}{1}$
- C. $\binom{4}{3}$
- ~~D. 3~~

$$(x + y)^4 = (x + y)(x + y)(x + y)(x + y)$$
$$= \underline{xxxx + yyyy + xyxy + yxyy + \dots}$$

Binomial Theorem: Idea

$$\binom{4}{1} xy^3 = 4x^1y^3$$

$$(x + y)^n = (x + y)(x + y)(x + y) \cdots (x + y)$$

Each term is of the form $x^k y^{n-k}$, since each term is made by multiplying exactly n variables, either x or y .

$$\binom{4}{1} = 4$$

$$xyyy + \dots + yxyy + ybxy + yyyx$$

How many times do we get $x^k y^{n-k}$? The number of ways to choose k of the n variables we multiple to be an x (the rest will be y).

$$n=2, k=1$$

$$x^1 y^{2-1}$$

$$(x+y)^2 = 1x^2 + 2xy + y^2$$

$$\binom{n}{k} = \binom{n}{n-k}$$

$$x^2 y^{2-2} = x^2$$

$$\binom{2}{2} = 1$$

$$\binom{2}{1} = 2$$

Binomial Theorem

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

Corollary.

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

Agenda

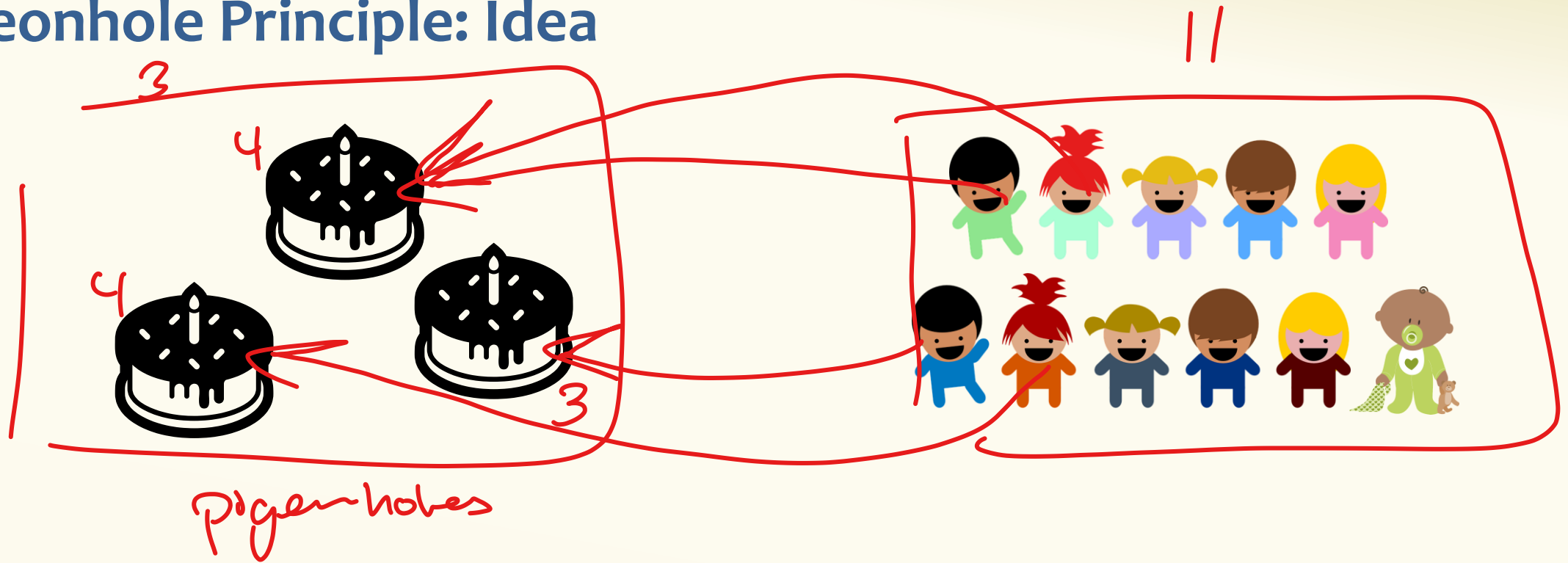
- Recap
- Inclusion-Exclusion
- Binomial Theorem
- **Pigeonhole Principle** ◀
- Combinatorial Proofs

Pigeonhole Principle (PHP): Idea

10 pigeons, 9 pigeonholes



Pigeonhole Principle: Idea



If 11 children have to share 3 cakes, at least one cake must be shared by how many children?

Pigeonhole Principle – More generally

$$n = 10 \quad h = 9 \quad \frac{10}{9} = 1.\overline{1} > 1$$

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\left(\frac{n}{k}\right)$ pigeons!

$$n = 11 \quad h = 3 \quad \frac{11}{3} = 3.\overline{6} > 3$$

Proof. Assume there are $< \left(\frac{n}{k}\right)$ pigeons per hole.

Then, there are $< \left[k \frac{n}{k} \right] = n$ pigeons overall.

Contradiction!

Pigeonhole Principle – Better version

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\lceil \frac{n}{k} \rceil$ pigeons!

Reason. Can't have fractional number of pigeons

Syntax reminder:

$\lceil x \rceil$

- Ceiling: $\lceil x \rceil$ is x rounded up to the nearest integer (e.g., $\lceil 2.731 \rceil = 3$)
- Floor: $\lfloor x \rfloor$ is x rounded down to the nearest integer (e.g., $\lfloor 2.731 \rfloor = 2$)

Pigeonhole Principle – Example

In a room with 367 people, there are at least two with the same birthday.

$$h = 365 < 367 = n$$

Solution:

1. **367** pigeons = people
2. **365** holes = possible birthdays
3. Person goes into hole corresponding to own birthday
4. By PHP, there must be two people with the same birthday

$$\left\lceil \frac{n}{h} \right\rceil = \left\lceil \frac{367}{365} \right\rceil = 2$$

Pigeonhole Principle: Strategy

To use the PHP to solve a problem, there are generally 4 steps

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

Pigeonhole Principle – Example (Surprising?)

*In every set S of 100 integers, there are at least **two** elements whose difference is a multiple of 37.*

absolu

When solving a PHP problem:

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

Pigeons: The integers, $n = 100$

Pigeonholes: $0 - 36$, $k = 37$

Mapping: The integer mod 37

PHP: There must be at least one value $0 - 36$ with $\lceil 100/37 \rceil = 3$ integers assigned to it.

This means there are at least 2 integers a and b where

$$a \% 37 = b \% 37$$

$$a - b = 0 \% 37$$

which means that the difference between them is a multiple of 37

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- **Combinatorial Proofs** ◀

Combinatorial proof: Show that $M = N$

- Let S be a set of objects
- Show how to count $|S|$ one way $\Rightarrow |S| = M$
- Show how to count $|S|$ another way $\Rightarrow |S| = N$
- Conclude that $M = N$

Binomial Coefficient – Many interesting and useful properties

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\binom{n}{n} = 1$$

$$\binom{n}{1} = n$$

$$\binom{n}{0} = 1$$

Fact. $\binom{n}{k} = \binom{n}{n-k}$

Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Pascal's Identity

Fact. $\sum_{k=0}^n \binom{n}{k} = 2^n$

Follows from Binomial theorem

Pascal's Identities

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

How to prove Pascal's identity?

Algebraic argument:

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} \\ &= 20 \text{ years later ...} \\ &= \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k}\end{aligned}$$

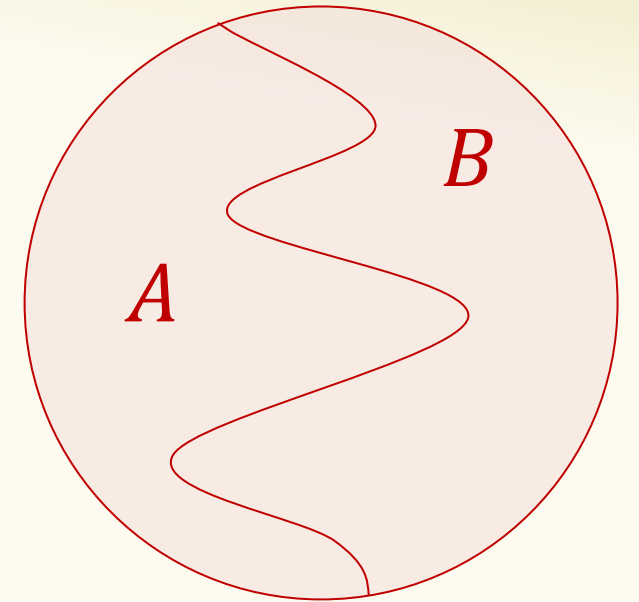
Hard work and not intuitive

Let's see a combinatorial argument

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| = |A| + |B|$



$S = A \cup B$, disjoint

S : the set of size k subsets of $[n] = \{1, 2, \dots, n\} \Rightarrow |S| = \binom{n}{k}$

A : the set of size k subsets of $[n]$ including n

B : the set of size k subsets of $[n]$ NOT including n

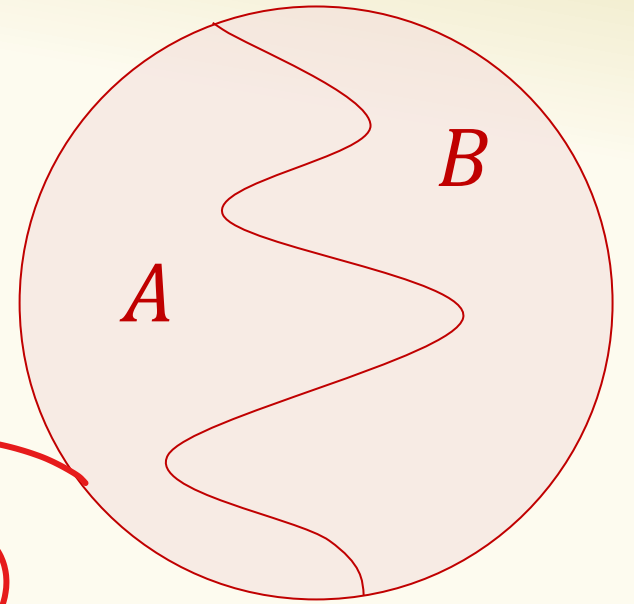
Sum rule:

$$|A \cup B| = |A| + |B|$$

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| = |A| + |B|$



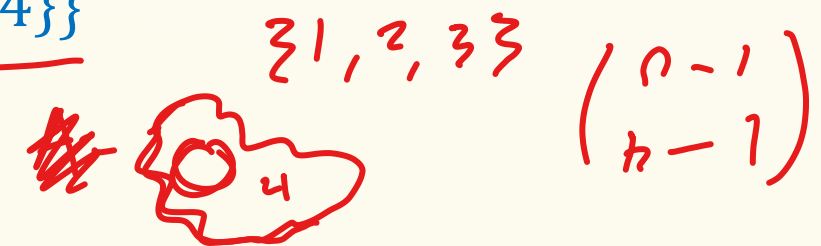
$\{1, 2, 3, 4\}$

S : the set of size k subsets of $[n] = \{1, 2, \dots, n\} \rightarrow |S| = \binom{n}{k}$

e.g.: $n = 4, k = 2, S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

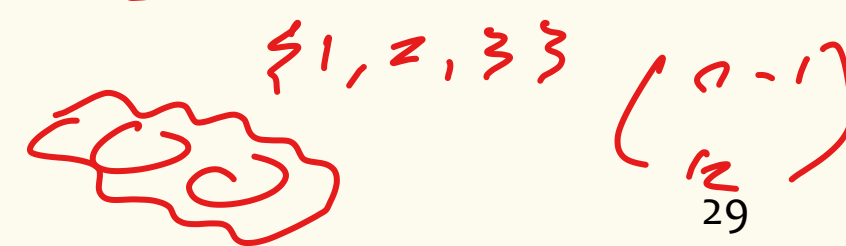
A : the set of size 2 subsets of $[4]$ including 4

$A = \{\{1,4\}, \{2,4\}, \{3,4\}\}$.



B : the set of size 2 subsets of $[4]$ NOT including 4

$B = \{\{1,2\}, \{1,3\}, \{2,3\}\}$



Example – Binomial Identity

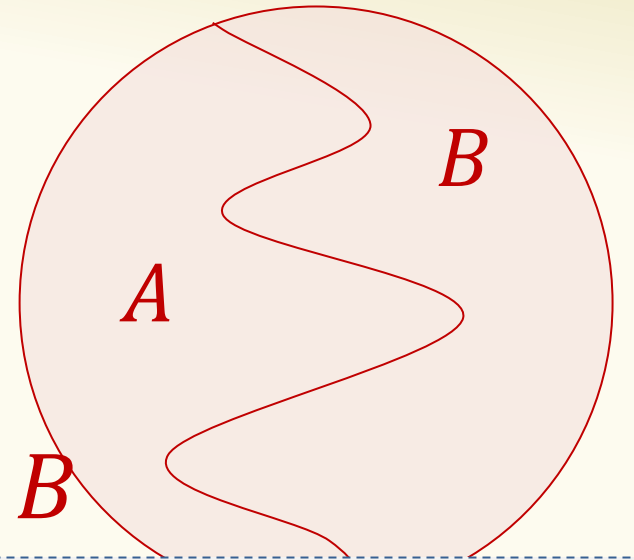
Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S|$

$|A|$

$|B|$

$S = A \cup B$



S : the set of size k subsets of $[n] = \{1, 2, \dots, n\}$

A : the set of size k subsets of $[n]$ including n

B : the set of size k subsets of $[n]$ NOT including n

n is in set, need to choose $k - 1$ elements from $[n - 1]$

$$|A| = \binom{n-1}{k-1}$$

n not in set, need to choose k elements from $[n - 1]$

$$|B| = \binom{n-1}{k}$$

combinatorial argument/proof

- Elegant
- Simple
- Intuitive



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Algebraic argument

- Brute force
- Less Intuitive



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Counting Recap

- **Core Theorems**
 - Sum & Product Rule
 - Permutations and Combinations
 - Inclusion-Exclusion
 - Binomial Theorem
- **Counting Strategies**
 - Complimentary Counting
 - Stars and Bars
 - Pigeonhole Principle
 - Combinatorial Proofs