CSE 312

Foundations of Computing II

Lecture 5: Conditional Probability Introduction

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

Belonging and CS Tas Research Study

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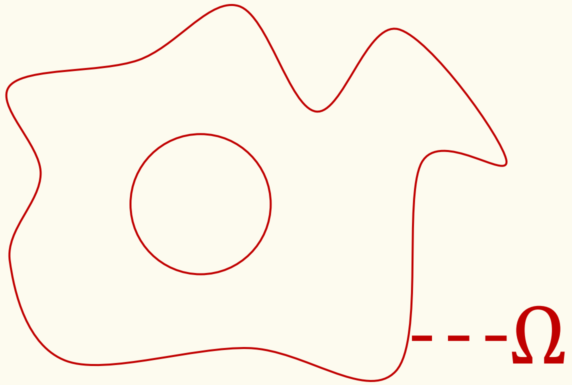
Review Probability space

Either finite or infinite countable (e.g., integers)

Definition. A (discrete) **probability space** is a pair $(\Omega, \mathbb{P})$ where:

- Ω is a set called the **sample space.**
- $\mathbb{P}$ is the **probability measure,** a function $\mathbb{P}: \Omega \to [0,1]$ such that:
  - $\mathbb{P}(\omega) \geq 0$ for all $\omega \in \Omega$
  - $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

Set of possible **elementary outcomes**



Specify Likelihood (or probability) of each **elementary outcome**

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Review Axioms of Probability

Let Ω denote the sample space and $E, F \subseteq \Omega$ be events. Note this is more general to **any** probability space (not just uniform)

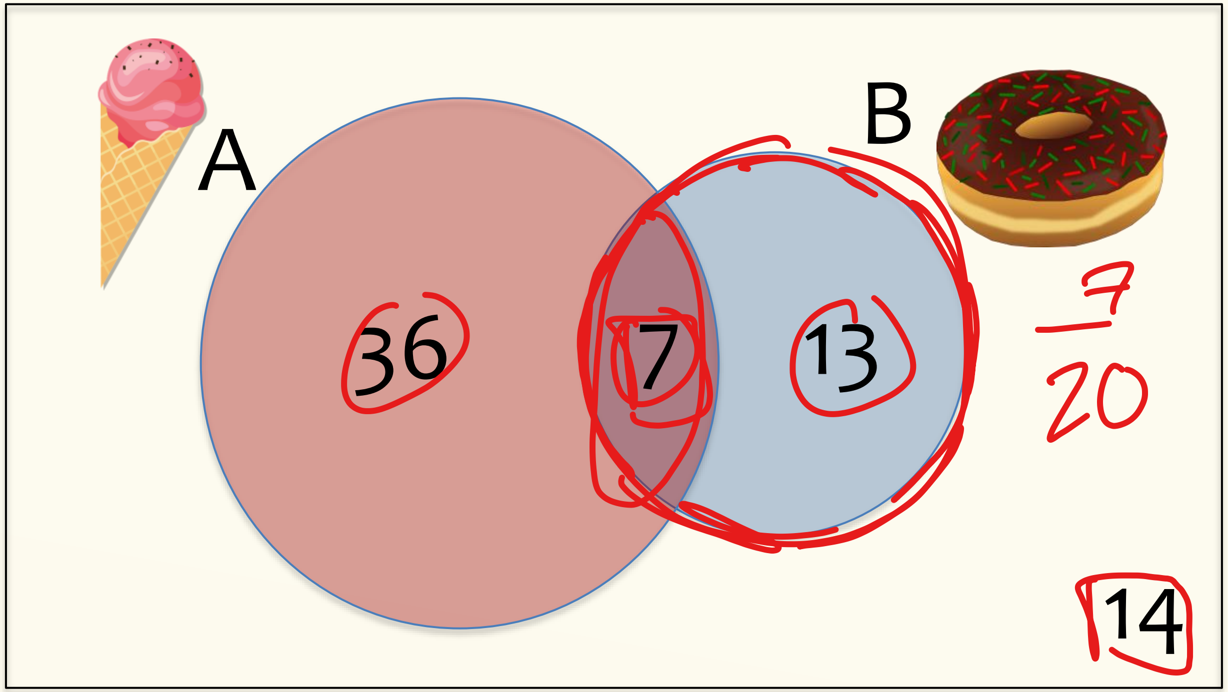
Axiom 1 (Non-negativity): $P(E) \geq 0$
Axiom 2 (Normalization): $P(\Omega) = 1$
Axiom 3 (Countable Additivity): If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$

Corollary 1 (Complementation): $P(E^c) = 1 - P(E)$
Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$
Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Agenda

- Conditional Probability
- Time Permitting:
  - Bayes Theorem
  - Law of Total Probability
  - Bayes Theorem + Law of Total Probability
  - More Examples

Conditional Probability (Idea)



What’s the probability that someone likes ice cream **given** they like donuts?

Conditional Probability

Definition. The **conditional probability** of event A **given** an event B happened (assuming $P(B) \neq 0$) is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$\mathbb{P}(A|B)$

An equivalent and useful formula is

A = likes ice cream

B = donuts ♡

$\mathbb{P}(B \cap A) = \mathbb{P}(B|A)\mathbb{P}(A)$ $\quad P(A \cap B) = \frac{7}{20} \cdot \frac{20}{70} = \frac{7}{70}$

$$P(A \cap B) = P(A|B)P(B)$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{7/70}{20/70} = \frac{7}{20}$$

Reversing Conditional Probability

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Question: Does $P(A|B) \stackrel{?}{=} P(B|A)$?

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

No! The following is purely for intuition and makes no sense in terms of probability

- Let A be the event you are wet
- Let B be the event you are swimming

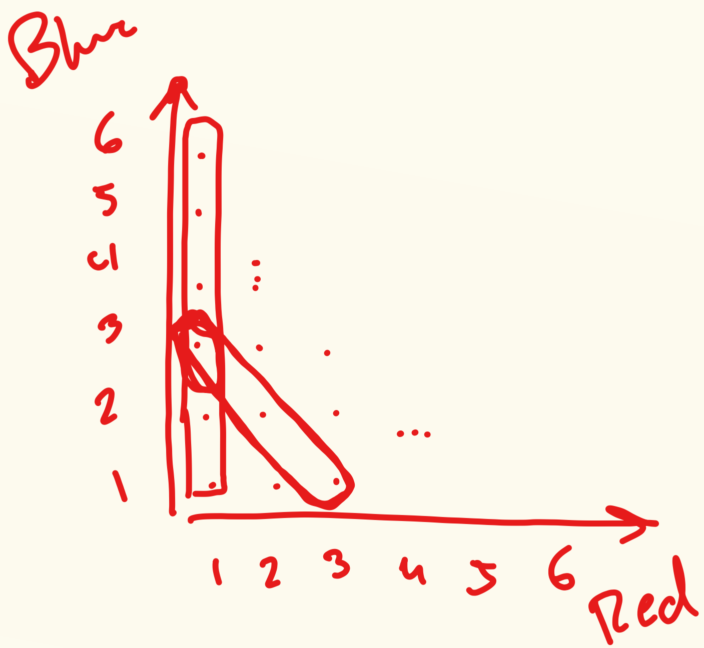
$$P(A|B) = 1$$
$$P(B|A) \neq 1$$

Example with Conditional Probability

Toss a red die and a blue die (both 6 sided and all outcomes equally likely). What is $P(B)$? What is $P(B|A)$?

	$P(B)$	$P(B\|A)$
a)	1/6	1/6
b)	1/6	1/3
c)	1/6	3/36
d)	1/9	1/3

Ω



B = red die is 1

A = sum is 4

$$\mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{6}{36} = \frac{1}{6}$$

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \frac{|B \cap A| / |\Omega|}{|A| / |\Omega|}$$

$$= \frac{1/36}{3/36} = \boxed{\frac{1}{3}}$$

Example with Conditional Probability

Toss a red die and a blue die (both 6 sided and all outcomes equally likely). What is $P(B)$? What is $P(B|A)$?

	$\mathrm{P}(B)$	$P(B\|A)$
a)	1/6	1/6
b)	1/6	1/3
c)	1/6	3/36
d)	1/9	1/3

B = red die is 1

A = sum is 4

$$\mathbb{P}(B) = \frac{1}{6}$$

$$\mathbb{P}(B|A) = \frac{1}{3}$$

Gambler's fallacy

Assume we toss **51** fair coins.
Assume we have seen **50** coins, and they are all "tails".
What are the odds the **51**st coin is "heads"?

$|\Omega| = 2^{51}$

$\mathcal{A}$ = first 50 coins are "tails"

B = 51st coin is "heads"

$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \frac{|B \cap A|}{|A|} = \boxed{\frac{1}{2}} = \mathbb{P}(B)$$

Gambler's fallacy

Assume we toss **51** fair coins.
Assume we have seen **50** coins, and they are all "tails".
What are the odds the **51**st coin is "heads"?

$\mathcal{A}$ = first 50 coins are "tails"

B = 51st coin is "heads"

$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(\mathcal{A} \cap \mathcal{B})}{\mathbb{P}(\mathcal{A})} = \frac{1/2^{51}}{2/2^{51}} = \frac{1}{2}$$

51st coin is independent of outcomes of first 50 tosses!

Gambler's fallacy = Feels like it's time for " heads"!?

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Bayes Theorem

A formula to let us "reverse" the conditional.

Theorem. (Bayes Rule) For events A and B, where $P(A), P(B) > 0$,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$P(A)$ is called the **prior** (our belief without knowing anything)

$P(A|B)$ is called the **posterior** (our belief after learning B)

Bayes Theorem Proof

By definition of conditional probability

$$P(A \cap B) = P(A|B)P(B)$$

Swapping A, B gives

$$P(B \cap A) = P(B|A)P(A)$$

But $P(A \cap B) = P(B \cap A)$, so

$$P(A|B)P(B) = P(B|A)P(A)$$

Dividing both sides by $P(B)$ gives

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Our First Machine Learning Task: Spam Filtering

Subject: “FREE $$$ CLICK HERE”

S F

What is the probability this email is spam, given the subject contains “FREE”?

Some useful stats:

- 10% of ham (i.e., not spam) emails contain the word “FREE” in the subject.
- 70% of spam emails contain the word “FREE” in the subject.
- 80% of emails you receive are spam.

$$P(F \mid S^c) = 0.1$$
$$P(F \mid S) = 0.7$$
$$P(S) = 0.8$$

$$P(S \mid F) = \frac{P(F \mid S)\,P(S)}{P(F)} = \frac{0.7 \cdot 0.8}{\quad}$$

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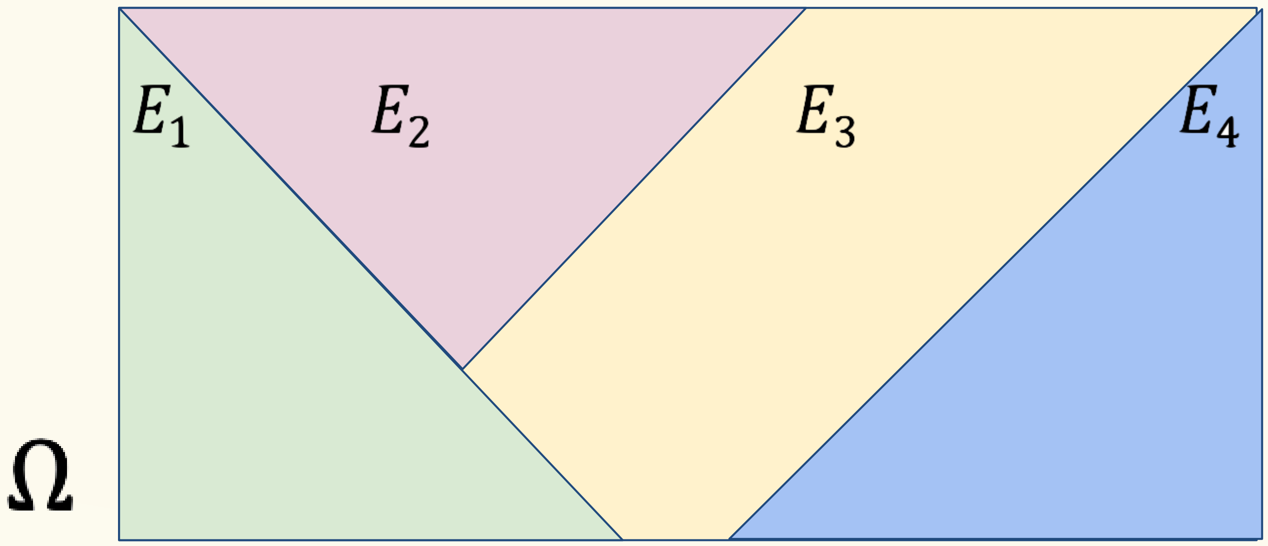
Partitions (Idea)

These events **partition** the sample space

1. They "cover" the whole space
2. They don't overlap



Partition

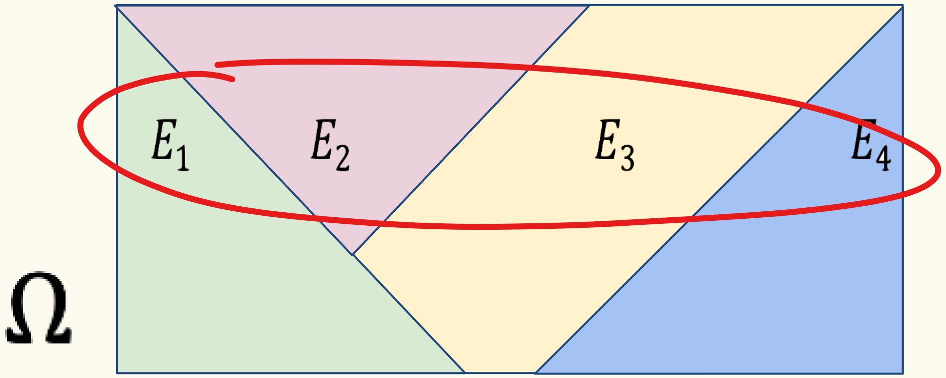
Definition. Non-empty events $E_1, E_2, \ldots, E_n$ **partition** the sample space Ω if

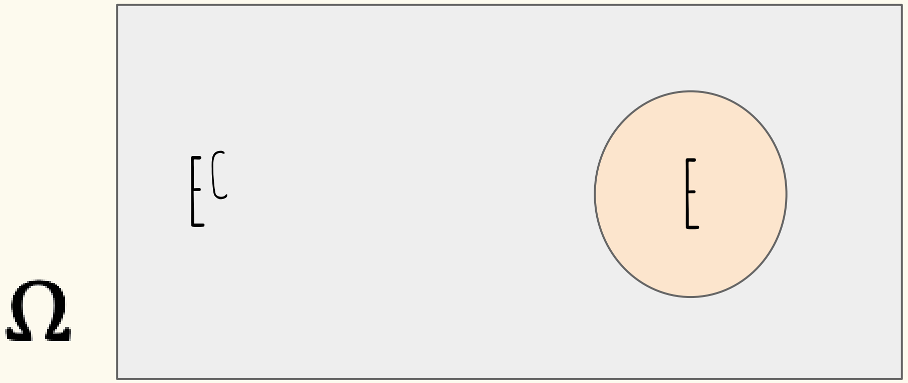
(Exhaustive)

$$E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$$

(Pairwise Mutually Exclusive)

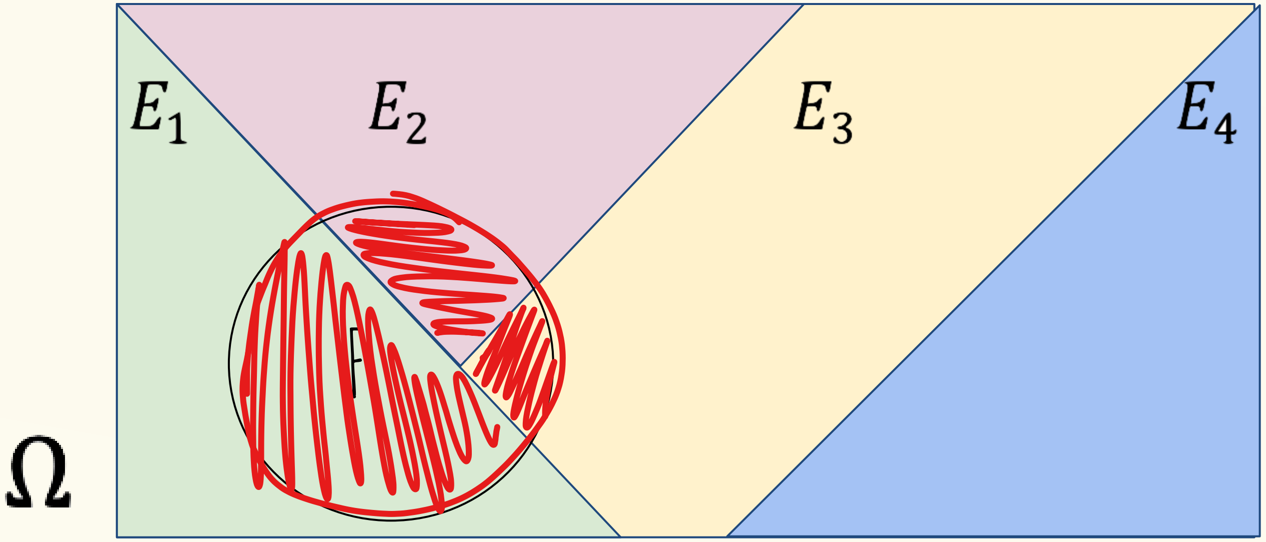
$$\forall_i \forall_{i \neq j} E_i \cap E_j = \emptyset$$





Law of Total Probability (Idea)

If we know $E_1, E_2, \ldots, E_n$ partition Ω, what can we say about $P(F)$



Law of Total Probability (LTP)

Definition. If events $E_1, E_2, \ldots, E_n$ partition the sample space Ω, then for any event F

$$P(F) = P(F \cap E_1) + \ \ldots + P(F \cap E_n) = \sum_{i=1}^{n} P(F \cap E_i)$$

Using the definition of conditional probability $P(F \cap E) = P(F|E)P(E)$

We can get the alternate form of this that show

$$P(F) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$

Another Contrived Example

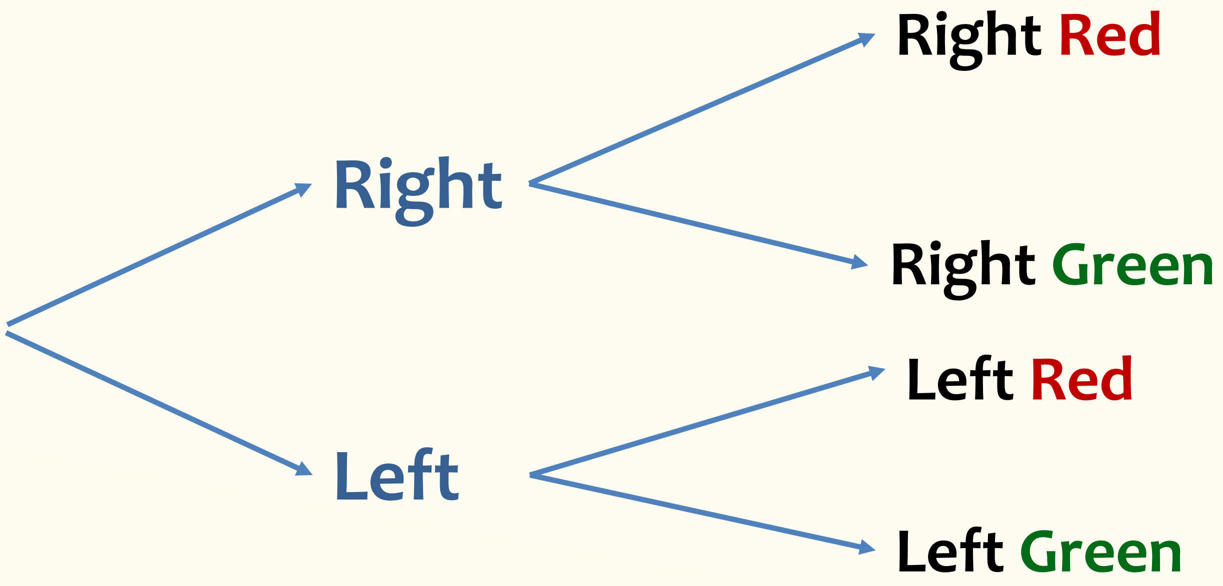
Alice has two pockets:

- **Left pocket:** Two red balls, two green balls
- **Right pocket:** One red ball, two green balls.

Alice picks a random ball from a random pocket.
[Both pockets equally likely, each ball equally likely.]

What is $\mathbb{P}(\mathbf{R})$?

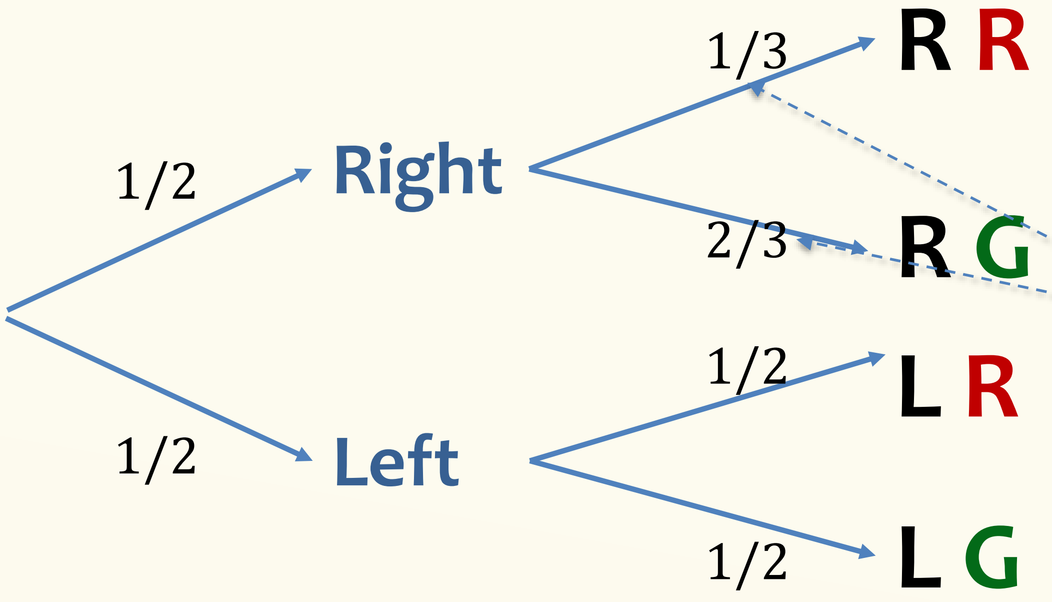
Sequential Process – Non-Uniform Case



- **Left pocket:** Two red, two green
- **Right pocket:** One red, two green.

- Alice picks a random ball from a random pocket

Sequential Process – Non-Uniform Case



- **Left pocket:** Two red, two green
- **Right pocket:** One red, two green.

1/3 = $\mathcal{P}(R \mid \mathrm{R})$ and 2/3 = $\mathcal{P}(G \mid \mathrm{R})$

$$\mathbb{P}(\mathbf{R}) = \mathbb{P}(\mathbf{R} \cap \mathbf{Left}) + \mathbb{P}(\mathbf{R} \cap \mathbf{Right})$$

(Law of total probability)

$$= \mathbb{P}(\mathbf{Left}) \times \mathbb{P}(\mathbf{R}|\mathbf{Left}) + \mathbb{P}(\mathbf{Right}) \times \mathbb{P}(\mathbf{R}|\mathbf{Right})$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

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What is the probability this email is spam, given the subject contains “FREE”?
Some useful stats:

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Bayes Theorem with Law of Total Probability

Bayes Theorem with LTP: Let $E_1, E_2, \dots, E_n$ be a partition of the sample space, and F and event. Then,

$$P(E_1|F) = \frac{P(F|E_1)P(E_1)}{P(F)} = \frac{P(F|E_1)P(E_1)}{\sum_{i=1}^{n} P(F|E_i)P(E_i)}$$

Simple Partition: In particular, if E is an event with non-zero probability, then

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^C)P(E^C)}$$

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Example – Zika Testing

Usually no or mild symptoms (rash); sometimes severe symptoms (paralysis).

During pregnancy: may cause birth defects.

Suppose you took a Zika test, and it returns "positive", what is the likelihood that you actually have the disease?

- Tests for diseases are rarely 100% accurate.

This example and following slides are from Lisa Yan (Stanford).

Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive")
- However, the test may yield a "false positive" 1% of the time
- 0.5% of the US population has Zika.

What is the probability you have Zika (event Z) if you test positive (event T).

A) Less than 0.25
B) Between 0.25 and 0.5
C) Between 0.5 and 0.75
D) Between 0.75 and 1

Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive")
- However, the test may yield a "false positive" 1% of the time
- 0.5% of the US population has Zika.

What is the probability you have Zika (event Z) if you test positive (event T).

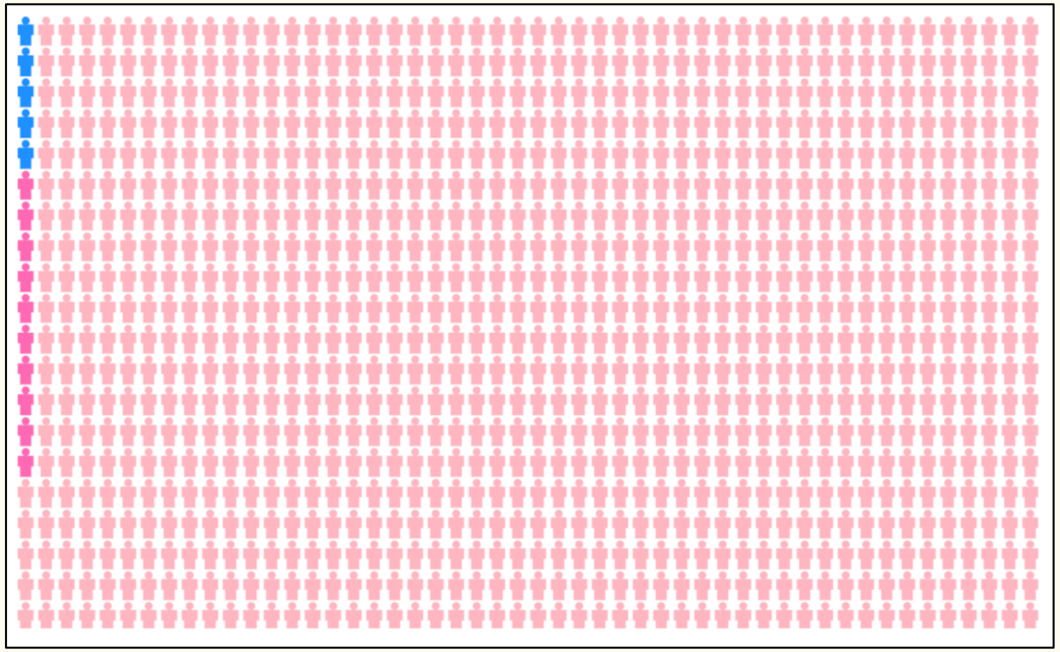
Example – Zika Testing

Have zika blue, don't pink

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive")
- However, the test may yield a "false positive" 1% of the time
- 0.5% of the US population has Zika. 5% have it.

What is the probability you have Zika (event Z) if you test positive (event T).



Suppose we had 1000 people:

- 5 **have Zika** and **test positive**
- 985 **do not have Zika** and **test negative**
- 10 **do not have Zika** and **test positive**

$$\frac{5}{5+10} = \frac{1}{3} \approx 0.33$$

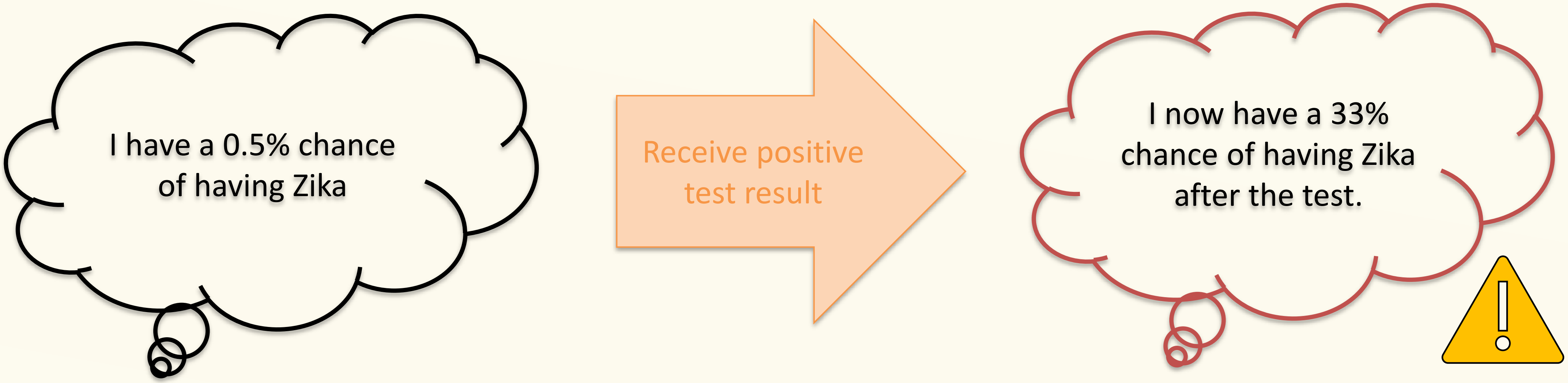
Demo

Philosophy – Updating Beliefs

While it's not 98% that you have the disease, your beliefs changed **drastically**

Z = you have Zika

T = you test positive for Zika



Prior: P(Z)

Posterior: P(Z|T)

Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive")
- However, the test may yield a "false positive" 1% of the time
- 0.5% of the US population has Zika.

What is the probability you test negative (event $\bar{T}$) if you have Zika (event Z)?

Conditional Probability Define a Probability Space

The probability conditioned on A follows the same properties as (unconditional) probability.

Example. $\mathbb{P}(\mathcal{B}^c|\mathcal{A}) = 1 - \mathbb{P}(\mathcal{B}|\mathcal{A})$

Conditional Probability Define a Probability Space

The probability conditioned on A follows the same properties as (unconditional) probability.

Example. $\mathbb{P}(\mathcal{B}^c|\mathcal{A}) = 1 - \mathbb{P}(\mathcal{B}|\mathcal{A})$

Formally. $(\Omega, \mathbb{P})$ is a probability space + $\mathbb{P}(\mathcal{A}) > 0$

⇒ $(\mathcal{A}, \mathbb{P}(\cdot\,|\mathcal{A}))$ is a probability space