CSE 312

Foundations of Computing II

Lecture 8: Introduction to Random Variables

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & Anna ☺

Agenda

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation

Random Variables (Idea)

Often: We want to **capture quantitative properties** of the outcome of a random experiment, e.g.:

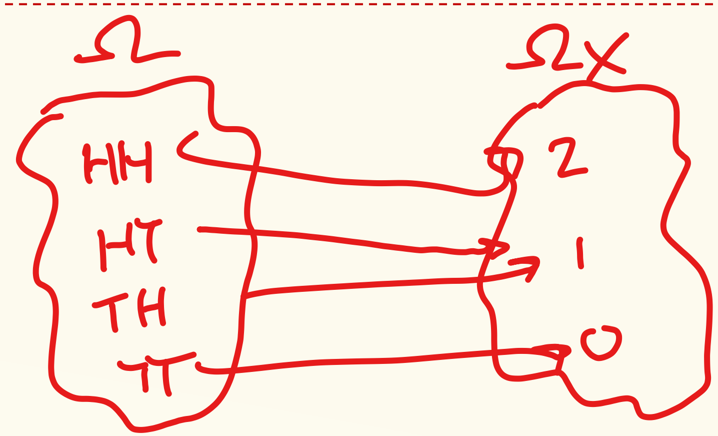
- *What is the total of two dice rolls?*
- *What is the number of coin tosses needed to see the first head?*
- *What is the number of heads among 2 coin tosses?*

Random Variables

Definition. A **random variable (RV)** for a probability space $(\Omega, \mathbb{P})$ is a function $X: \Omega \to \mathbb{R}$.

The set of values that X can take on is called its range/support Ω_X

Example. Number of heads in 2 independent coin flips $\Omega = \{HH, HT, TH, TT\}$



RV Example

$\binom{20}{3} = |\Omega|$

20 balls labeled 1, 2, …, 20 in a bin

- Draw a subset of 3 uniformly at random
- Let $X = \text{maximum of the 3 numbers on the balls}$
  - Example: $X(2,7,5) = 7$
  - Example: $X(15,3,8) = 15$
- What is $|\Omega_X|$?

$|\Omega_X| = 18$

A. 20^3
B. 20
C. 18
~~D.~~ $\cancel{\binom{20}{3}}$

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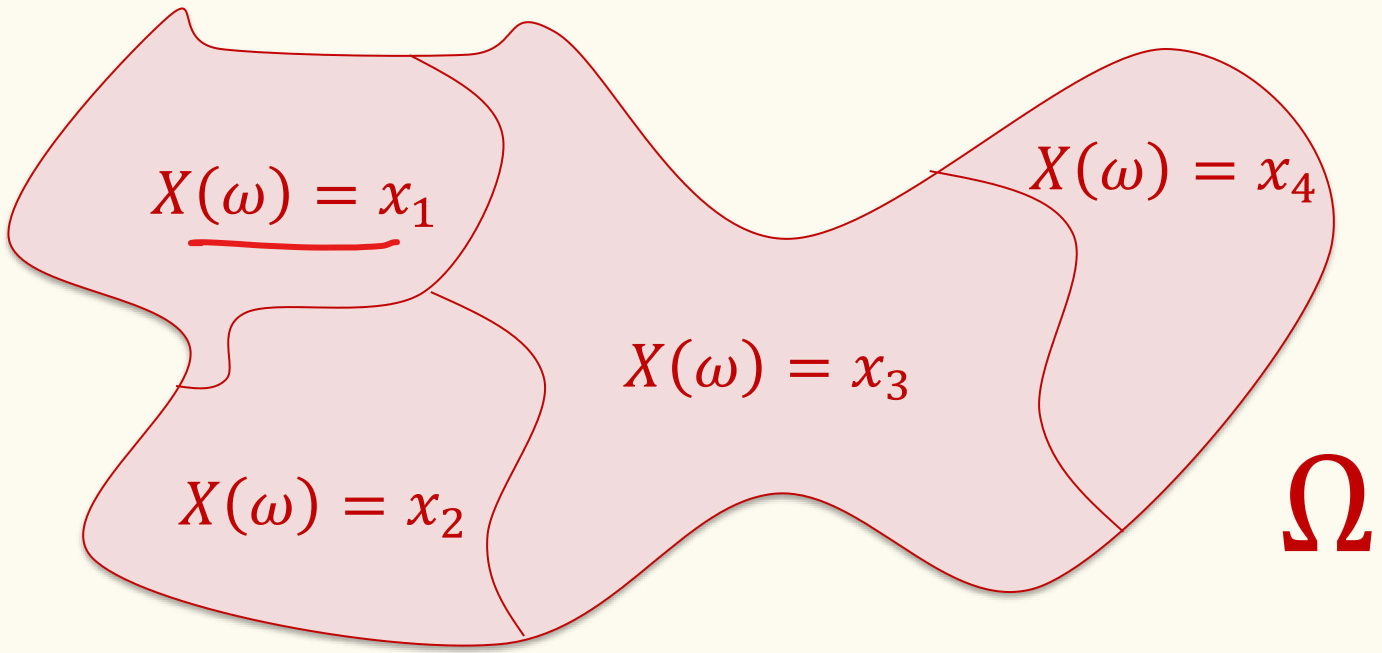
Probability Mass Function (PMF)

$\{HT, TH\}$

$\frac{|E|}{|\Omega|}$

Random variables partition the sample space.

$$\sum_{x \in X(\Omega)} \mathbb{P}(X = x) = 1$$



Probability Mass Function (PMF)

Definition. For a RV $X: \Omega \to \mathbb{R}$, we define the event

$$\{X = x\} \stackrel{\text{def}}{=} \{\omega \in \Omega \mid X(\omega) = x\}$$

We write $\mathbb{P}(X = x) = \mathbb{P}(\{X = x\}) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$ where $\mathbb{P}(X = x)$ is the probability mass function (PMF) of X

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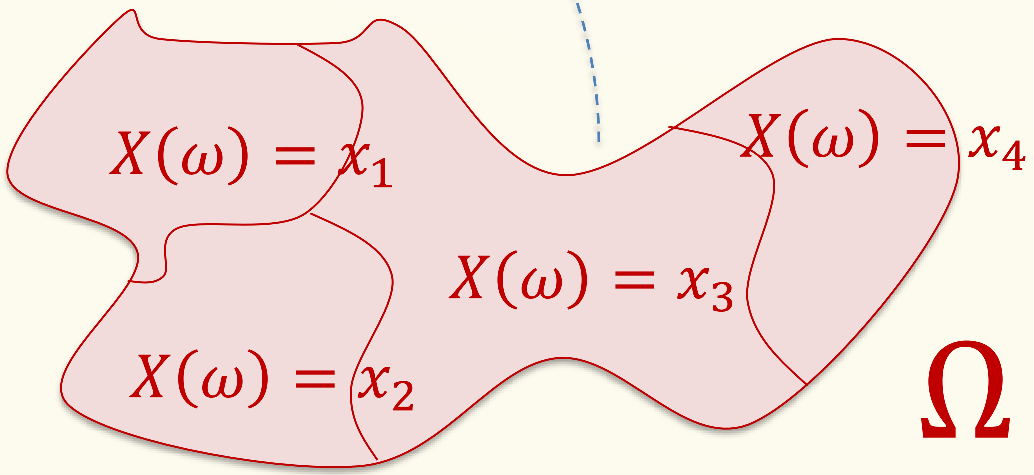

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Random variables partition the sample space.

$$\sum_{x \in X(\Omega)} \mathbb{P}(X = x) = 1$$

You also see this notation (e.g. in book):

$$\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}) = \boldsymbol{p_X(x)}$$

Probability Mass Function

$\frac{|E|}{|\Omega|} = \frac{1}{4}$

Flipping two independent fair coins

$\Omega = \{HH, HT, TH, TT\}$

X = number of heads in the two flips

$X(HH) = 2 \qquad X(HT) = 1 \qquad X(TH) = 1 \qquad X(TT) = 0$

$\mathbb{P}(X = 0)$

$\Omega_X = \{0, 1, 2\}$

What is $Pr(X = k)$?

$$\mathbb{P}(X = k) = \begin{cases} 1/4 & k = 0 \\ 1/2 & k = 1 \\ 1/4 & k = 2 \end{cases}$$

RV Example

Ω

20 balls labeled 1, 2, …, 20 in a bin

- Draw a subset of 3 uniformly at random
- Let X = maximum of the 3 numbers on the balls

$3 \ldots 20$

What is $\mathbb{P}(X = 20)$?

$$\frac{|X=20|}{|\Omega|} = \boxed{\frac{\binom{19}{2}}{\binom{20}{3}}}$$

A. $\binom{20}{2} / \binom{20}{3}$

B. $\binom{19}{2} / \binom{20}{3}$

C. $19^2 / \binom{20}{3}$

D. $19 \cdot 18 / \binom{20}{3}$

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Cumulative Distribution Function (CDF)

$F_X(0.5) = P(X \le 0.5)$

Definition. For a RV $X: \Omega \to \mathbb{R}$, the cumulative distribution function of where X specifies for any real number x, the probability that $X \le x$.

$$\mathrm{F}_X(x) = \Pr(X \le x)$$

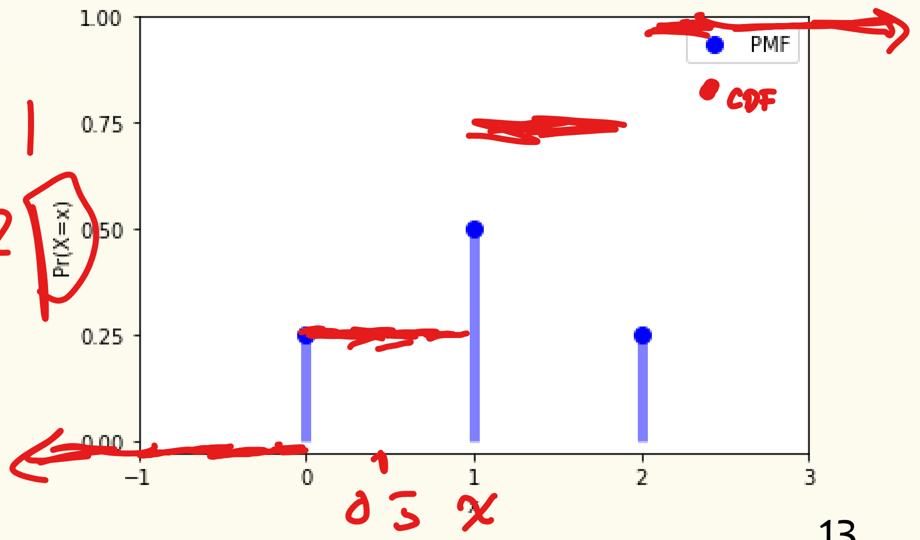
$P(X \le x) \quad x < 0$

$P(X \le 0) = P(X = 0) = \frac{1}{4}$

Go back to 2 coin clips, where X is the number of heads

$$\Pr(X = x) = \begin{cases} \frac{1}{4}, & x = 0 \\ \frac{1}{2}, & x = 1 \\ \frac{1}{4}, & x = 2 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1/4 & 0 \le x < 1 \\ 3/4 & 1 \le x < 2 \\ 1 & 2 \le X \end{cases}$$



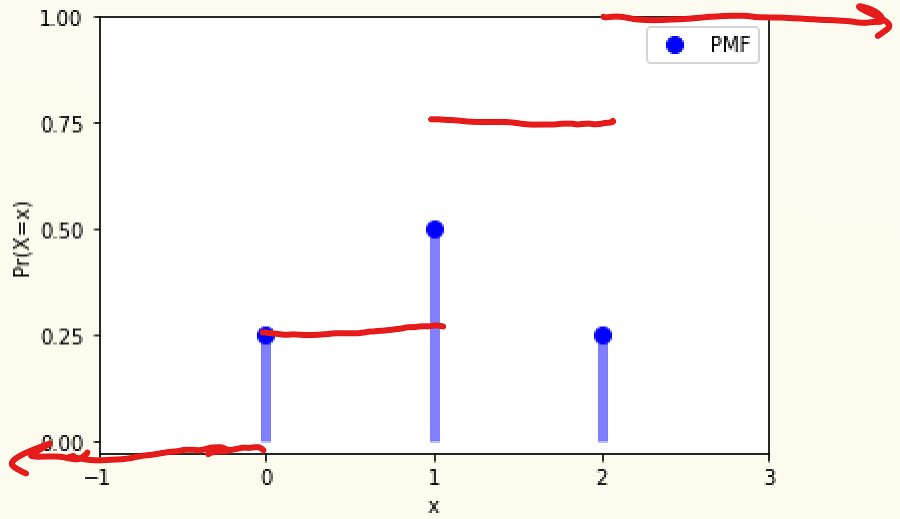
Cumulative Distribution Function (CDF)

Definition. For a RV $X: \Omega \to \mathbb{R}$, the cumulative distribution function of where X specifies for any real number x, the probability that $X \le x$.

$$\mathrm{F}_X(x) = \Pr(X \le x)$$

Go back to 2 coin clips, where X is the number of heads

$$\Pr(X = x) = \begin{cases} \frac{1}{4}, & x = 0 \\ \frac{1}{2}, & x = 1 \\ \frac{1}{4}, & x = 2 \end{cases} \qquad F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \le x < 1 \\ \frac{3}{4}, & 1 \le x < 2 \\ 1, & 2 \le x \end{cases}$$



Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$\Omega_X : \{0, 1, 3\}$

1 2 3

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

$$\mathbb{P}(X = x) : \begin{cases} 1/3 & x = 0 \\ 1/2 & x = 1 \\ 1/6 & x = 3 \end{cases}$$

$$F_X(x) : \mathbb{P}(X \le x) = \begin{cases} 0 & x < 0 \\ 1/3 & 0 \le x < 1 \\ 5/6 & 1 \le x < 3 \\ 1 & 3 \le x \end{cases}$$

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Expectation (Idea)

What is the *expected* number of heads in 2 independent flips of a fair coin?

~~Cumulative Disribution Function (CDF)~~ Expectation

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the expectation or expected value of X is

$E[X]$

$$\mathrm{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega)$$

or equivalently

$$E[X] = \sum_{x \in \Omega_X} x \cdot \mathbb{P}(X = x) = 0 \cdot \mathbb{P}(X=0) \to 0 + 1 \cdot \mathbb{P}(X=1) \to 0.5 + 2 \cdot \mathbb{P}(X=2) \to 0.5 = \boxed{1}$$

$$\mathrm{E}[X] = \sum_{x \in X(\Omega)} x \cdot \Pr(X = x)$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

$$P(X=h) = \begin{cases} 1/3 & h=0 \\ 1/2 & h=1 \\ 1/6 & h=3 \end{cases}$$

$$E[X] = \sum_{x \in \Omega_X} x \cdot \mathbb{P}(X=x) = 0 \cdot \mathbb{P}(X=0) + 1 \cdot \mathbb{P}(X=1) + 3 \cdot \mathbb{P}(X=3)$$

$$\to 0 + 1/2 + 1/2 = \boxed{1}$$

Flip a Biased Coin Until Heads (Independent Flips)

Suppose a coin has probability p of being heads. Keep flipping independent flips until heads. Let X be the number of flips until heads.

What is: $\Pr(X = 1) =$

What is: $\Pr(X = 2) =$

What is: $\Pr(X = k) =$

Flip a Biased Coin Until Heads (Independent Flips)

Suppose a coin has probability p of being heads. Keep flipping independent flips until heads. Let X be the number of flips until heads. What is $\mathrm{E}[X]$?

Students on a bus

A group of 120 students are driven on 3 buses to a football game. There are 36 students in the first bus, 40 in the second and 44 in the third. Let Y be the number of students on a uniformly random bus. What is the pmf of Y and E(Y)? When the buses arrive, one of the 120 students is randomly chosen. Let X denote the number of students on the bus of the randomly chosen student. What is the pmf of X and what is E(X)?

Coin flipping again

Suppose we flip a coin with probability p of coming up Heads n times. Let X be the number of Heads in the n coin flips. What is the pmf of X?