

CSE 312

Foundations of Computing II


Lecture 13: Continuous RVs and the Exponential Distribution



Aleks Jovicic

Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself 😊

Agenda

- Continuous RVs 
 - Cumulative Distribution Function
 - Expectation and Variance
- Exponential Distribution
- Time permitting: Memorylessness

Definition. A **continuous random variable** X is defined by a **probability density function** (PDF) $f_X: \mathbb{R} \rightarrow \mathbb{R}$, such that

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

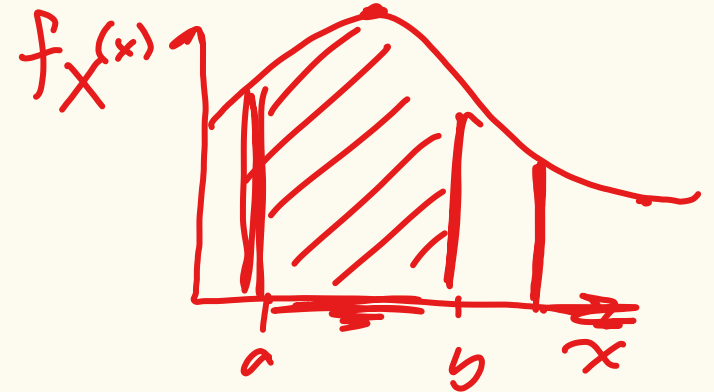
Normalization: $\int_{-\infty}^{+\infty} f_X(x) dx = 1$

$$\underline{P(a \leq X \leq b)} = \int_a^b f_X(x) dx$$


$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x) dx = 0$$

$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y - \frac{\epsilon}{2}}^{y + \frac{\epsilon}{2}} f_X(x) dx \approx \epsilon f_X(y)$$

$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$



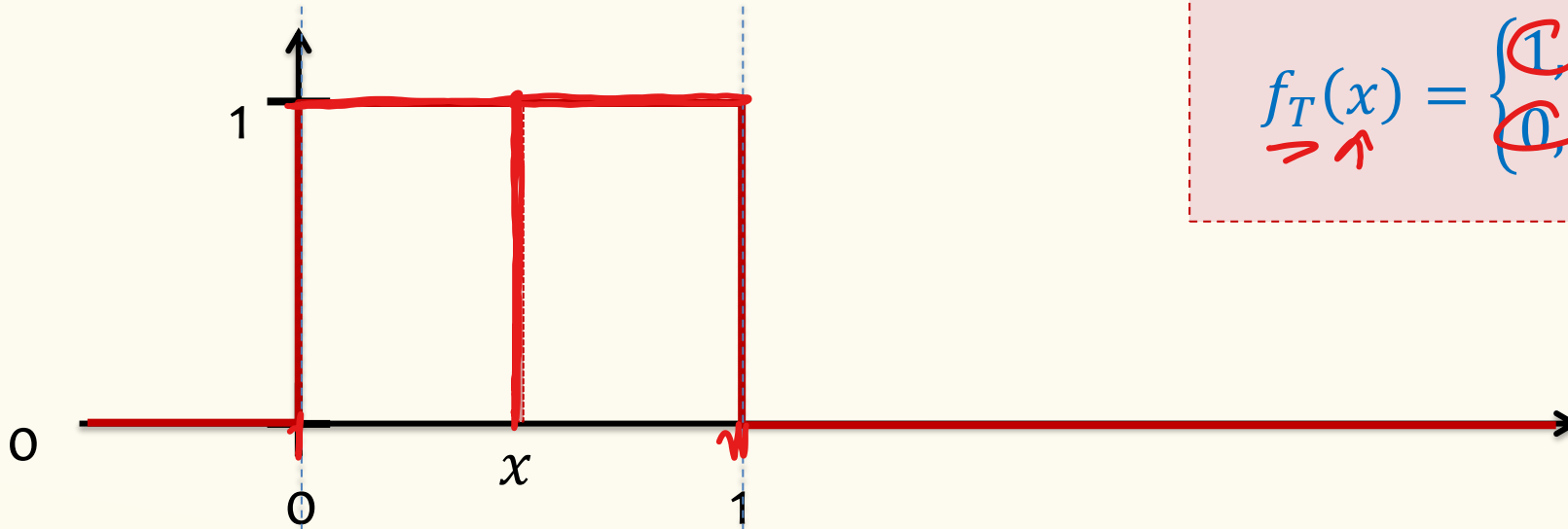
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Example. $T \sim \text{Unif}(0,1)$

Probability Density Function

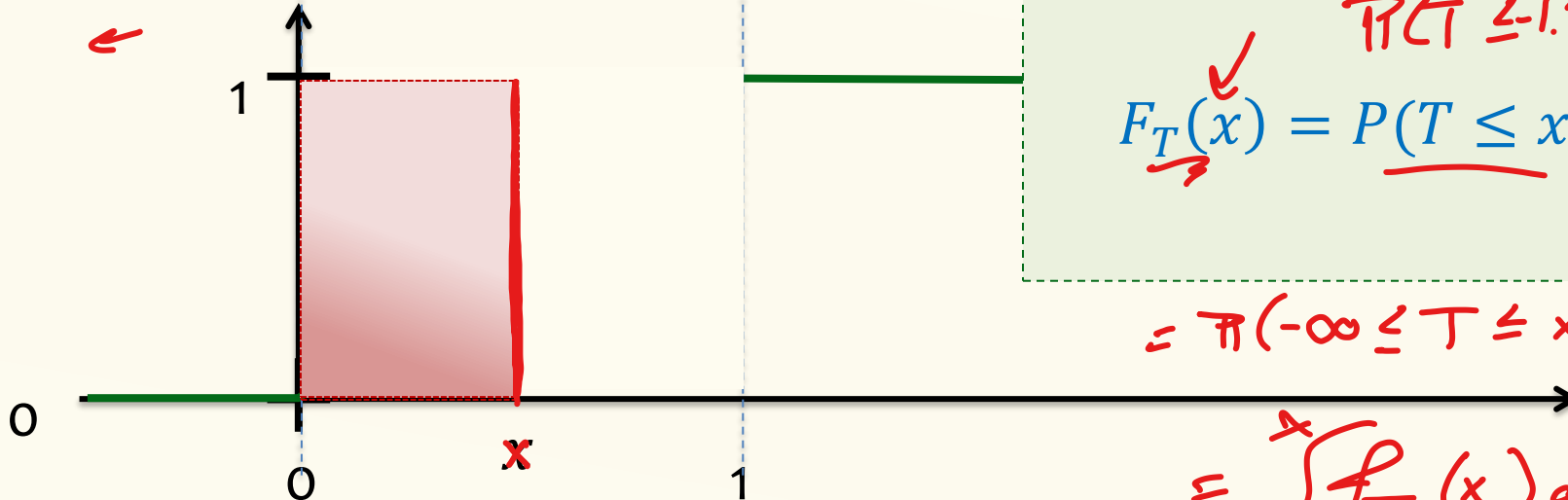
$$f_T(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$$



Cumulative Distribution Function

$$F_T(x) = P(T \leq x) = \begin{cases} 0 & x \leq 0 \\ ? & 0 \leq x \leq 1 \\ 1 & 1 < x \end{cases}$$

$-\infty$



$$= P(-\infty \leq T \leq x) = \int_{-\infty}^x f_T(x) dx$$

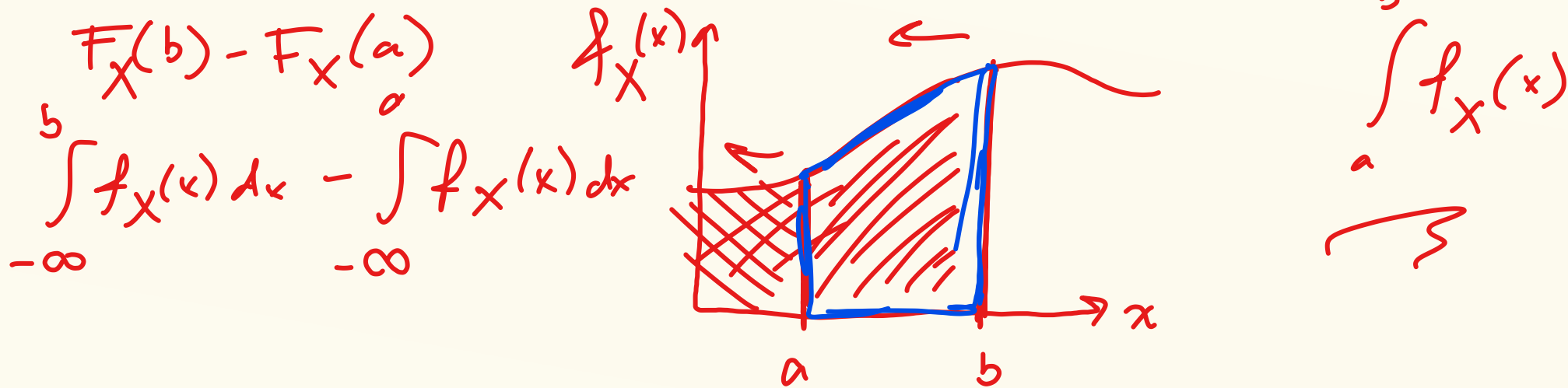
$$= \int_0^x f_T(x) dx$$

Cumulative Distribution Function

Definition. The **cumulative distribution function (cdf)** of X is

$$F_X(a) = \mathbb{P}(X \leq a) = \int_{-\infty}^a f_X(x) dx$$

By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx} F(x)$



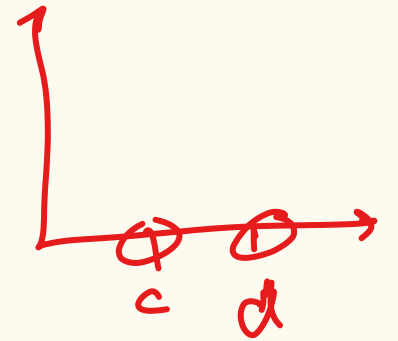
Cumulative Distribution Function

Definition. The **cumulative distribution function (cdf)** of X is

$$F_X(a) = \mathbb{P}(X \leq a) = \int_{-\infty}^a f_X(x) dx$$

By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx} F(x)$

Therefore: $\mathbb{P}(X \in [a, b]) = F(b) - F(a)$



F_X is monotone increasing, since $f_X(x) \geq 0$. That is $F_X(c) \leq F_X(d)$ for $c \leq d$

$$\lim_{a \rightarrow -\infty} F_X(a) = P(X \leq -\infty) = 0 \quad \lim_{a \rightarrow +\infty} F_X(a) = P(X \leq +\infty) = 1$$

From Discrete to Continuous

	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t < x} p_X(t)$	$F_X(x) = \int_{-\infty}^x f_X(t) dt$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x) dx = 1$
Expectation	$\mathbb{E}[g(X)] = \sum_x g(x)p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x) dx$

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Expectation of a Continuous RV

Definition. The **expected value** of a continuous RV X is defined as

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x dx$$

Fact. $\mathbb{E}(aX + bY + c) = a\mathbb{E}(X) + b\mathbb{E}(Y) + c$

$\mathbb{E}[g(x)]$

Definition. The **variance** of a continuous RV X is defined as

$$\text{Var}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot (x - \mathbb{E}(X))^2 dx = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

Uniform Distribution

$$X \sim \text{Unif}(a, b)$$

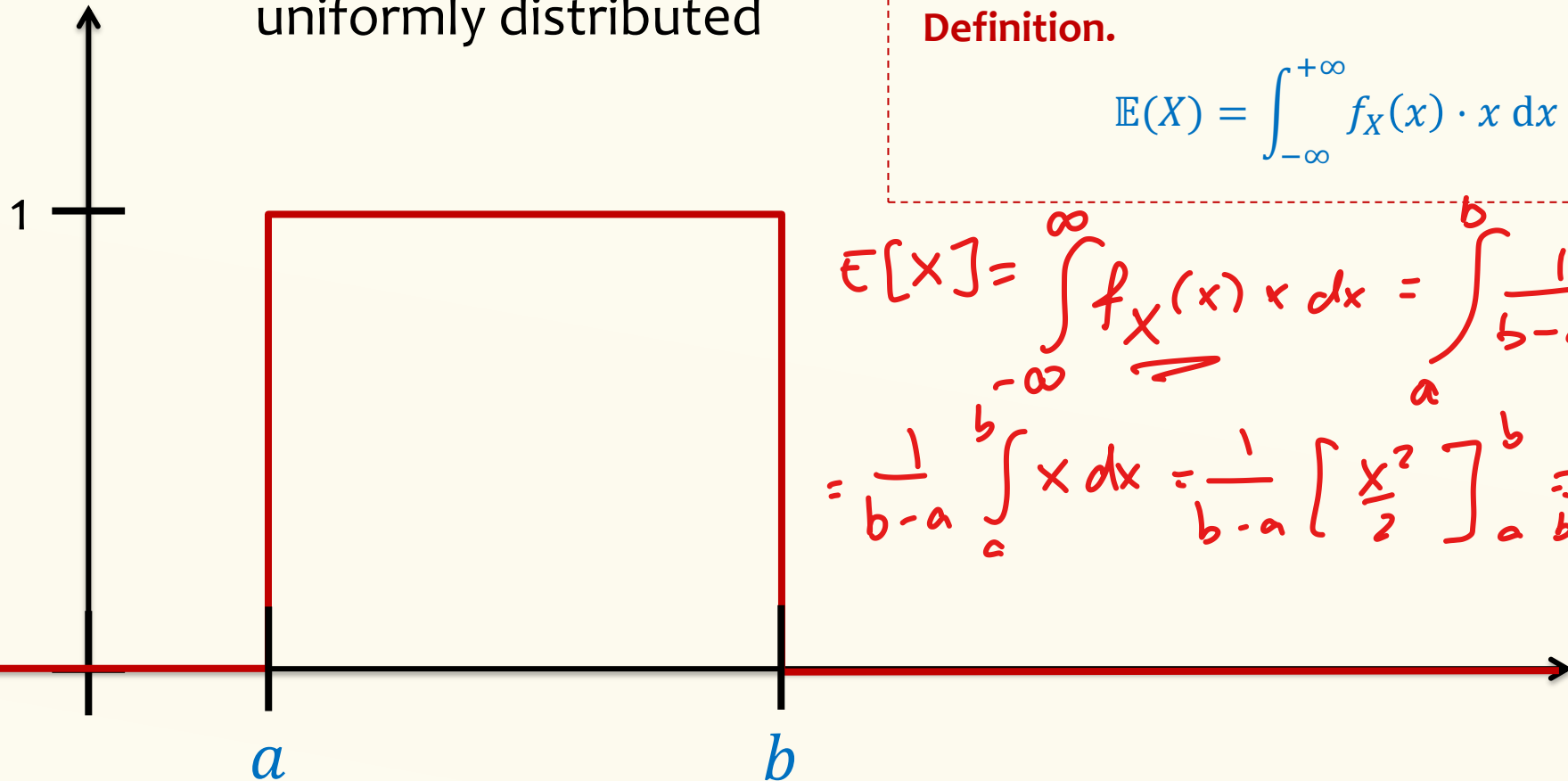
We also say that X follows the uniform distribution / is uniformly distributed

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

Definition.

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx$$

$\frac{b^2 - a^2}{(b-a)^2}$
 ~~$\frac{(b-a)(b+a)}{(b-a)^2}$~~
 $\frac{a+b}{2}$



$$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{\infty} f_X(x) \cdot x \, dx = \int_a^b \frac{1}{b-a} \cdot x \, dx \\ &= \frac{1}{b-a} \int_a^b x \, dx = \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b = \frac{1}{b-a} \left(\frac{b^2}{2} - \frac{a^2}{2} \right) \end{aligned}$$

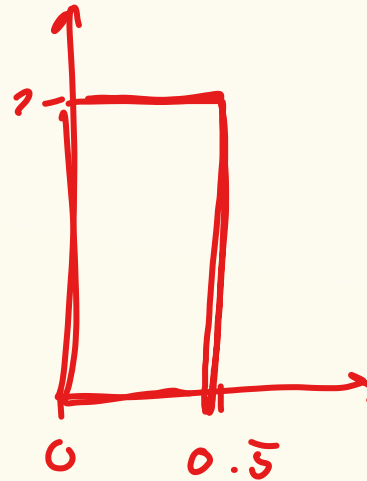
Uniform Density – Expectation

$$X \sim \text{Unif}(a, b)$$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx$$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$\frac{1}{0.5-0} = \frac{1}{0.5} = 2$$



Uniform Density – Expectation

$$X \sim \text{Unif}(a, b)$$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx$$

$$\begin{aligned} &= \frac{1}{b-a} \int_a^b x \, dx = \frac{1}{b-a} \left(\frac{x^2}{2} \right) \Big|_a^b = \frac{1}{b-a} \left(\frac{b^2 - a^2}{2} \right) \\ &= \frac{(b-a)(a+b)}{2(b-a)} = \frac{a+b}{2} \end{aligned}$$

Uniform Density – Variance

$$X \sim \text{Unif}(a, b)$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 \, dx$$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

Uniform Density – Variance

$$X \sim \text{Unif}(a, b)$$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 \, dx$$

$$= \frac{1}{b-a} \int_a^b x^2 \, dx = \frac{1}{b-a} \left(\frac{x^3}{3} \right) \Big|_a^b = \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

Uniform Density – Variance

$$\mathbb{E}(X^2) = \frac{b^2 + ab + a^2}{3}$$

$$\mathbb{E}(X) = \frac{a + b}{2}$$

$$X \sim \text{Unif}(a, b)$$

$$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4}$$

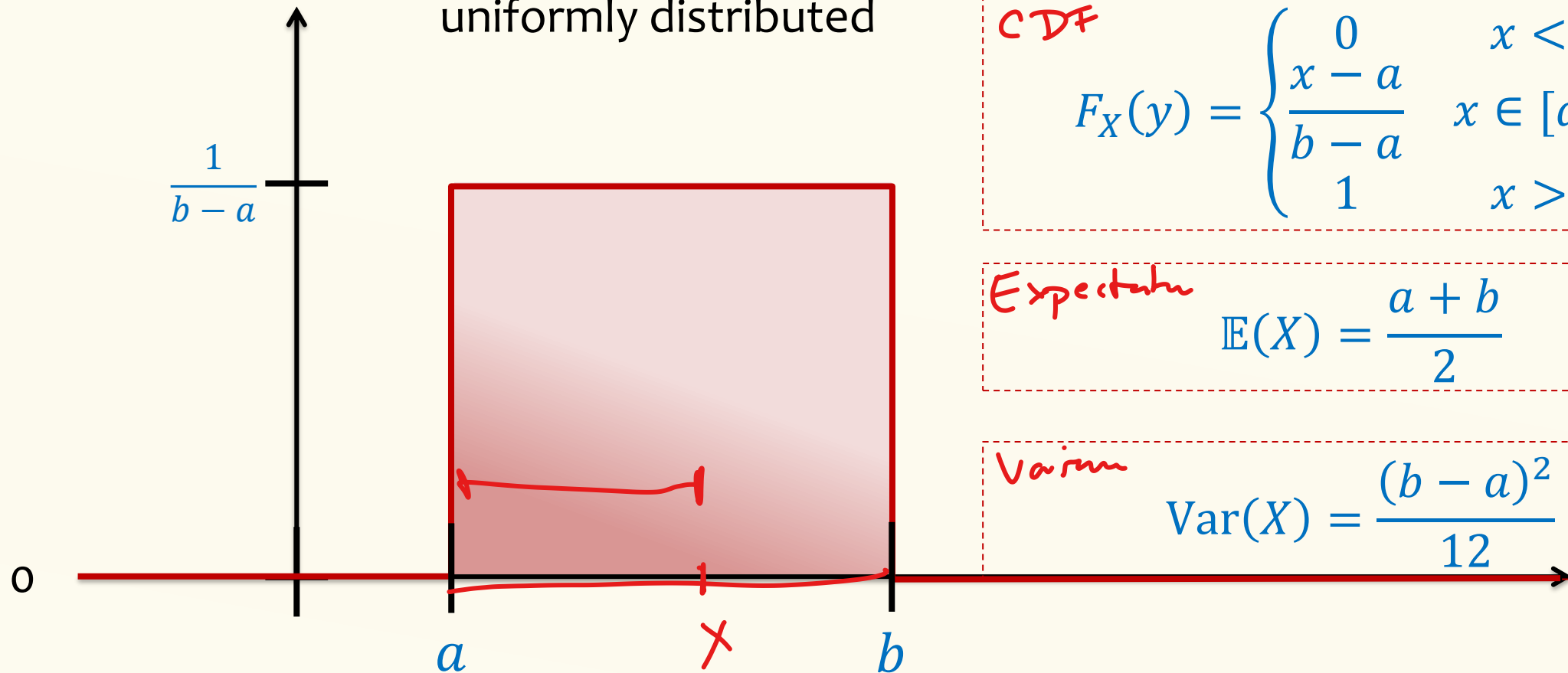
$$= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12}$$

$$= \frac{b^2 - 2ab + a^2}{12} = \frac{(b - a)^2}{12}$$

Uniform Distribution

$$X \sim \text{Unif}(a, b)$$

We also say that X follows the uniform distribution / is uniformly distributed



PDF

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

CDF

$$F_X(y) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & x \in [a, b] \\ 1 & x > b \end{cases}$$

Expectation

$$\mathbb{E}(X) = \frac{a+b}{2}$$

Variance

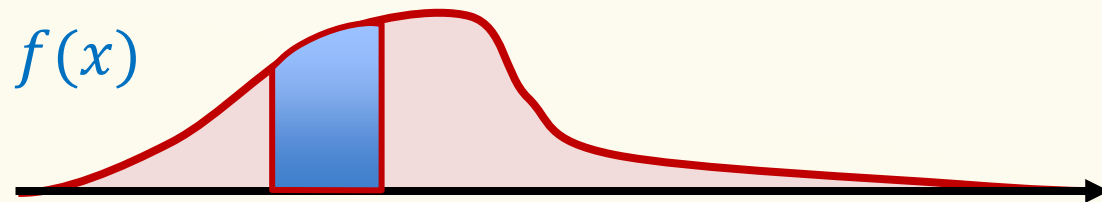
$$\text{Var}(X) = \frac{(b-a)^2}{12}$$

Review – Continuous RVs

Probability Density Function (PDF).

$f: \mathbb{R} \rightarrow \mathbb{R}$ s.t.

- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x) dx = 1$



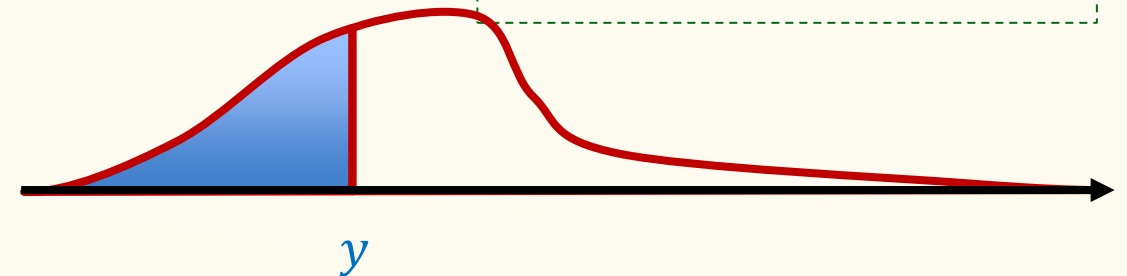
Density \neq Probability !

$$\begin{aligned}\mathbb{P}(X \in [a, b]) &= \int_a^b f_X(x) dx \\ &= F_X(b) - F_X(a)\end{aligned}$$

Cumulative Density Function (CDF).

$$F(y) = \int_{-\infty}^y f(x) dx$$

Theorem. $f(x) = \frac{dF(x)}{dx}$



$$F(y) = \mathbb{P}(X \leq y)$$

Expectation of a Continuous RV

Definition. The **expected value** of a continuous RV X is defined as

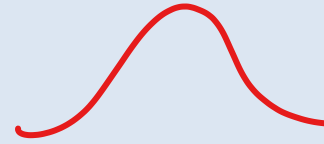
$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx$$

Fact. $\mathbb{E}(aX + bY + c) = a\mathbb{E}(X) + b\mathbb{E}(Y) + c$

Definition. The **variance** of a continuous RV X is defined as

$$\text{Var}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot (x - \mathbb{E}(X))^2 \, dx = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

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✓ Uniform
Exponential

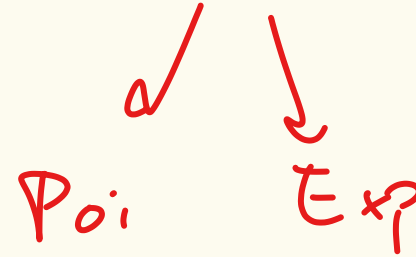
Normal
(Week 7)

Exponential Density

Assume expected # of occurrences of an event per unit of time is λ

- Cars going through intersection
- Number of lightning strikes
- Requests to web server
- Patients admitted to ER

Poi Exp



Numbers of occurrences of event: Poisson distribution

$$\mathbb{P}(X = i) = e^{-\lambda} \frac{\lambda^i}{i!} \quad (\text{Discrete})$$

How long to wait until next event? Exponential density!

Let's define it and then derive it!

The Exponential PDF/CDF

Assume expected # of occurrences of an event per unit of time is λ

Numbers of occurrences of event: Poisson distribution

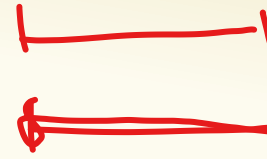
How long to wait until next event? Exponential density!

- The exponential RV has range $[0, \infty]$, unlike Poisson with range $\{0, 1, 2, \dots\}$
- Let $Y \sim \text{Exp}(\lambda)$ be the time till the first event. We will compute $F_Y(t)$ and $f_Y(t)$

$$Y \sim \text{Exp}(\lambda)$$

The Exponential PDF/CDF

$$\lambda = 5$$
$$Poi(5)$$



$$3 \cdot 5 = 15$$
$$Poi(15)$$

Assume expected # of occurrences of an event per unit of time is λ

Numbers of occurrences of event: Poisson distribution

How long to wait until next event? Exponential density!

- The exponential RV has range $[0, \infty]$, unlike Poisson with range $\{0, 1, 2, \dots\}$
- Let $Y \sim Exp(\lambda)$ be the time till the first event. We will compute $F_Y(t)$ and $f_Y(t)$
- Let $X \sim Poi(t\lambda)$ be the # of events in the first t units of time, for $t \geq 0$.

$$P(Y > t) = P(\text{no event in the first } t \text{ units}) = P(X = 0) = e^{-t\lambda} \frac{t\lambda^0}{0!} = e^{-t\lambda}$$

$$F_Y(t) = 1 - P(Y > t) = 1 - e^{-t\lambda}$$

$$f_Y(t) = \frac{d}{dt} F_Y(t) = \lambda e^{-t\lambda}$$

Exponential Distribution

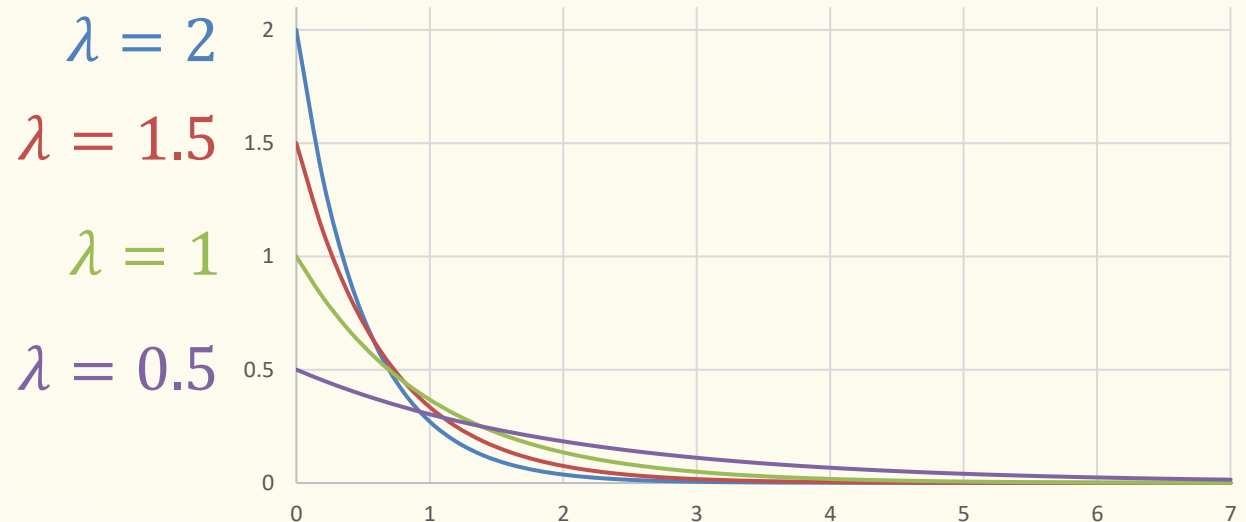
Definition. An **exponential random variable** X with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write $X \sim \text{Exp}(\lambda)$ and say X that follows the exponential distribution.

CDF: For $y \geq 0$,

$$F_X(y) = 1 - e^{-\lambda y}$$



Expectation

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx$$

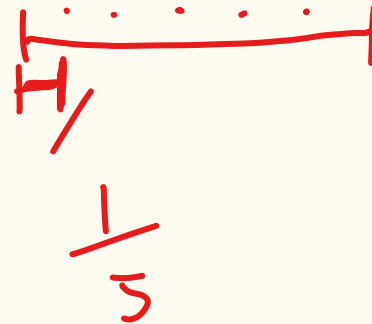
$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Expectation

$$\lambda = 5$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\begin{aligned} \mathbb{E}(X) &= \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx \\ &= \int_0^{+\infty} \lambda e^{-\lambda x} \cdot x \, dx \\ &= \left(-\left(x + \frac{1}{\lambda}\right) e^{-\lambda x} \right) \Big|_0^{\infty} = \frac{1}{\lambda} \end{aligned}$$



$$\mathbb{E}(X) = \frac{1}{\lambda}$$
$$\text{Var}(X) = \frac{1}{\lambda^2}$$

Somewhat complex calculation
use integral by parts

Example

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins.

$$X \sim \text{Exp}\left(\frac{1}{10}\right)$$

$$E[X] = 10 = \frac{1}{\lambda}$$

$$\mathbb{P}(10 \leq X \leq 20) = F_X(20) - F_X(10) =$$

Example

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins?

$$T \sim \text{Exp}\left(\frac{1}{10}\right)$$

$$P(10 \leq T \leq 20) = \int_{10}^{20} \frac{1}{10} e^{-\frac{x}{10}} dx$$

$$y = \frac{x}{10}, dy = \frac{dx}{10}$$

$$P(10 \leq T \leq 20) = \int_1^2 e^{-y} dy = -e^{-y} \Big|_1^2 = \boxed{e^{-1} - e^{-2}}$$

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Memorylessness

Definition. A random variable is **memoryless** if for all $s, t > 0$,

$$\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t).$$

Fact. $X \sim \text{Exp}(\lambda)$ is memoryless.

Assuming exp distr, if you've waited s minutes,
prob of waiting t more is exactly same as $s = 0$

Memorylessness of Exponential

Assuming exp distr, if you've waited s minutes, prob of waiting t more is exactly same as $s = 0$

Fact. $X \sim \text{Exp}(\lambda)$ is memoryless.

Proof.

$$\mathbb{P}(X > s + t \mid X > s)$$

Assuming exp distr, if you've waited s minutes, prob of waiting t more is exactly same as $s = 0$

Memorylessness of Exponential

Fact. $X \sim \text{Exp}(\lambda)$ is memoryless.

Proof.

$$\begin{aligned}\mathbb{P}(X > s + t \mid X > s) &= \frac{\mathbb{P}(\{X > s + t\} \cap \{X > s\})}{\mathbb{P}(X > s)} \\ &= \frac{\mathbb{P}(X > s + t)}{\mathbb{P}(X > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathbb{P}(X > t)\end{aligned}$$

The only memoryless RVs are the geometric RV (discrete) and Exp RV (continuous)