CSE 312
Foundations of Computing II

Lecture 14: Joint Distributions

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Agenda

- Cartesian Products ◀
- Joint PMFs/PDFs/CDFs and Joint Range
- Independence
- Marginal Distributions
- Expectation
- Joint Continuous Random Variables

Why joint distributions?

- Given all of its user's ratings for different movies, and any preferences you have expressed, Netflix wants to recommend a new movie for you.

- Given a bunch of medical data correlating symptoms and personal history with diseases, predict what is ailing a person with a particular medical history and set of symptoms.

- Given current traffic, pedestrian locations, weather, lights, etc. decide whether a self-driving car should slow down or come to a stop.

Review Cartesian Product

(a, b)

Definition. Let A and B be sets. The **Cartesian product** of A and B is denoted

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Example.

A B $|A \times B| = 6$

$$\{1,2,3\} \times \{4, 5\} = \{(1, 4), (1,5), (2,4), (2,5), (3, 4), (3,5)\}$$

$|A| = 3$ $|B| = 2$

If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$.

$(\mathbb{R}, \mathbb{R})$

The sets don't need to be finite! You can have $\mathbb{R} \times \mathbb{R}$ (often denoted $\mathbb{R}^2$)

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Joint PMFs and Joint Range

$\Omega(X): \{1,2,3\}$

$\Omega(Y) \{4,5\}$

Definition. Let X and Y be discrete random variables. The **Joint PMF** of X and Y is

$$p_{X,Y}(a,b) = \Pr(X = a, Y = b)$$

Definition. The **joint range** of $p_{X,Y}$ is

$$\Omega(X,Y) = \{(c,d) : p_{X,Y}(c,d) > 0\} \subseteq \Omega(X) \times \Omega(Y)$$
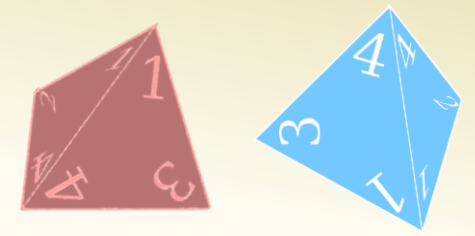
Note that

$$\sum_{(s,t)\in\Omega(X,Y)} p_{X,Y}(s,t) = 1$$

$(1,4) \quad (2,4) \quad (3,4)$
$(1,5) \quad (2,5) \quad (3,5)$

$(7,10)$

Example: Weird Dice

$p_{X,Y}(1,1) = P(X=1, Y=1)$
$= P(X=1)P(Y=1)$
$\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die.

$\Omega(X) = \{1,2,3,4\}$ and $\Omega(Y) = \{1,2,3,4\}$

In this problem, the joint PMF is

$$p_{X,Y}(x,y) = \begin{cases} 1/16, & x,y \in \Omega(X,Y) \\ 0, & \text{otherwise} \end{cases}$$

X\Y	1	2	3	4
1	1/16	1/16	1/16	1/16
2	1/16	1/16	1/16	1/16
3	1/16	1/16	1/16	1/16
4	1/16	1/16	1/16	1/16

and the joint range is (since all combinations have non-zero probability)
$\Omega(X,Y) = \Omega(X) \times \Omega(Y)$

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Independence

$$p_{X,Y}(x,y) = P(X = x, Y = y)$$
$$= P(X = x) \cdot P(Y = y)$$

Definition. Let X and Y be discrete random variables. The **Joint PMF** of X and Y is

$$p_{X,Y}(a,b) = \Pr(X = a, Y = b)$$

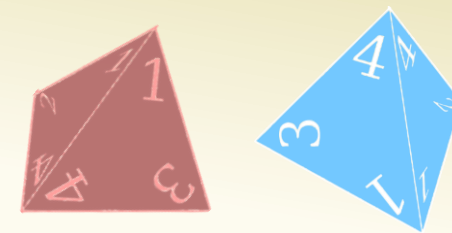**Definition.** The **joint range** of $p_{X,Y}$ is

$$\Omega(X,Y) = \{(c,d) : p_{X,Y}(c,d) > 0\} \subseteq \Omega(X) \times \Omega(Y)$$

Definition. X and Y are **independent** iff for all a, b

$$\Pr(X = a, Y = b) = \Pr(X = a) \cdot \Pr(Y = b) \qquad \Omega_{X,Y} = \Omega_X \times \Omega_Y$$

Example: Weirder Dice

$(1, 3)$

$(3, 1)$

Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

$\Omega(U) = \{1,2,3,4\}$ and $\Omega(W) = \{1,2,3,4\}$

$(4, 3)$

$\underset{U \quad W}{}$

$\Omega(U, W) = \{(u, w) \in \Omega(U) \times \Omega(W) : u \leq w\} \neq \Omega(U) \times \Omega(W)$

Poll:

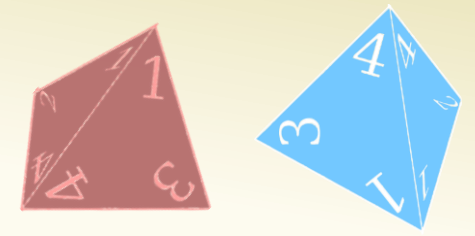What is $p_{U,W}(1, 3) = \Pr(U = 1, W = 3)$? $\Big\} = 2/16$

$= \dfrac{\Pr(X = 1, Y = 3)}{+ \Pr(X = 3, Y = 1)} \begin{array}{l} 1/16 \\ = 1/16 \end{array}$

a. 1/16

b. 2/16

c. 1/2

d. Not sure

U\W	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

Example: Weirder Dice

Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

$\Omega(U) = \{1,2,3,4\}$ and $\Omega(W) = \{1,2,3,4\}$

$\Omega(U, W) = \{(u, w) \in \Omega(U) \times \Omega(W) : u \leq w\} \neq \Omega(U) \times \Omega(W)$
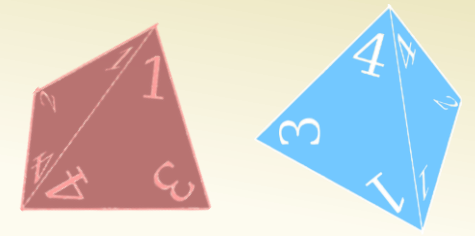
The joint PMF $p_{U,W}(u, w) = \Pr(U = u, W = w)$ is

$$p_{U,W}(u, w) = \begin{cases} 2/16, & (u, w) \in \Omega(U) \times \Omega(W) \quad \text{where } w > u \\ 1/16, & (u, w) \in \Omega(U) \times \Omega(W) \quad \text{where } w = u \\ 0, & \text{otherwise} \end{cases}$$

U\W	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

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- **Marginal Distributions** ◀
- Expectation
- Joint Continuous Random Variables

Example: Weirder Dice

Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

Suppose we didn't know how to compute $\Pr(U = u)$ directly. Can we figure it out if we know $p_{U,W}(u, w)$?

$$\Pr(U = 2) = \sum_{w=1} \Pr(U = 2 \cap W = w)$$

$$p_U(u) = \begin{cases} 7/16 & u = 1 \\ 5/16 & u = 2 \\ 3/16 & u = 3 \\ 1/16 & u = 4 \end{cases}$$

$= \Pr(U = 2 \cap W = 1)$

$+ \Pr(U = 2 \cap W = 2)$

$\cdots$

$+ \Pr(U = 2 \cap W = 4)$

U\W	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

Example: Weirder Dice
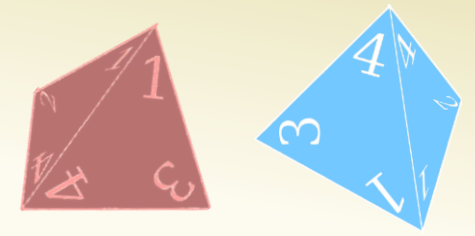
Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

Suppose we didn't know how to compute $\Pr(U = u)$ directly. Can we figure it out if we know $p_{U,W}(u, w)$?

$$p_U(u) = \begin{cases} 7/16, & u = 1 \\ 5/16, & u = 2 \\ 3/16, & u = 3 \\ 1/16, & u = 4 \end{cases}$$

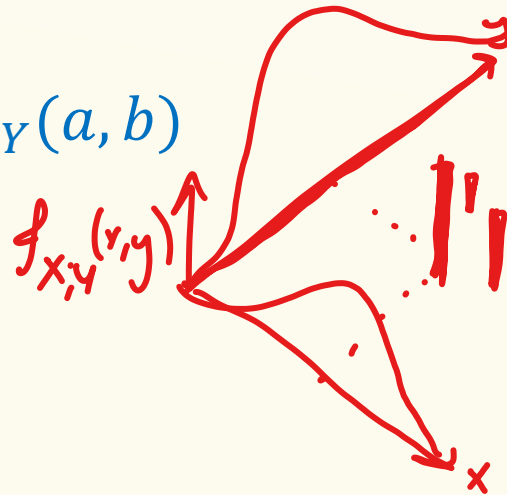$$\mathbb{P}_U(3) = \sum_{w \in \Omega(u)} p_{U,w}(3, w)$$
$$= p_{U,w}(3, 1) + \ldots$$

U\W	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

Marginal PMF

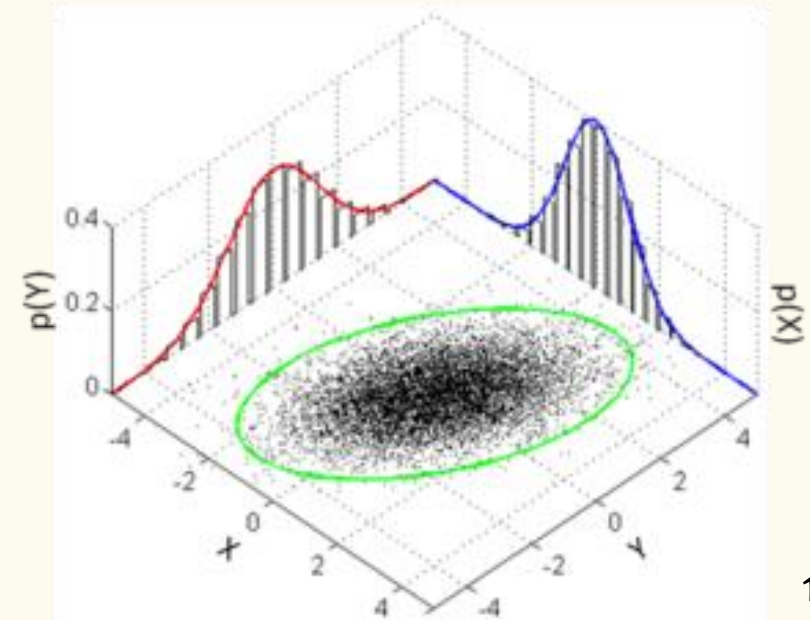**Definition.** Let X and Y be discrete random variables and $p_{X,Y}(a, b)$ their joint PMF. The **marginal PMF** of X
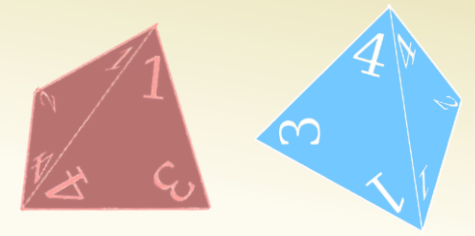
$$p_X(a) = \sum_{b \in \Omega(Y)} p_{X,Y}(a, b)$$

Similarly, $p_Y(b) = \sum_{a \in \Omega(X)} p_{X,Y}(a, b)$

$f_{X,Y}(x,y)$

Visual (for continuous X and Y)

Marginal PMF Example

Suppose the table below gives us the joint pmf of X and Y.

What is the marginal pmf of X? What is the marginal pmf of Y?
Are X and Y independent?

$$P_X(x) = \sum_{y \in \Omega_y} P_{X,Y}(x,y)$$

$$P_X(z) = P_{X,Y}(1,1) + P_{X,Y}(2,2)$$
$$0.1 \qquad 0 - 4$$
$$= 0.5$$

$$P_x(1) = P_{X,Y}(1,1) + P_{X,Y}(1,2)$$

$$0.5$$

X\Y	1	2
1	0.4	0.1
2	0.1	0.4

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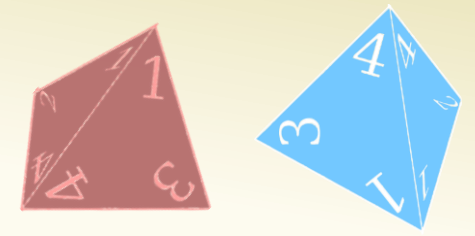- Expectation ◄
- Joint Continuous Random Variables

Joint Expectation

Definition. Let X and Y be discrete random variables and $p_{X,Y}(a,b)$ their joint PMF. The **expectation** of some function $g(x,y)$ with inputs X and Y

$$E[g(X,Y)] = \sum_{a \in \Omega(X)} \sum_{b \in \Omega(Y)} g(a,b) p_{X,Y}(a,b)$$

$$E[g(X)] = \sum_{r \in \Omega_X} g(x) \cdot \mathbb{P}(X = x)$$

Expectation Example

Suppose the table below gives us the joint pmf of X and Y.

$$g(x,y) = xy$$

What is E(XY)?

$$E[XY] = \sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} \underset{xy}{\cancel{g(x,y)}} P_{X,Y}(x,y)$$

$$= 1 \cdot 1 \cdot 0.4$$
$$+ 1 \cdot 2 \cdot 0.1$$
$$+ 2 \cdot 1 \cdot 0.1$$
$$+ 2 \cdot 2 \cdot 0.4$$

$$= 0.4 + 0.2 + 0.2 + 1.6$$

$$= \boxed{2.4}$$

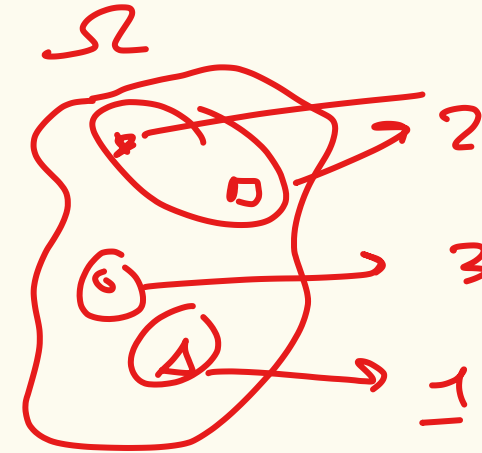| X|Y	1	2
1	0.4	0.1
2	0.1	0.4

- Suppose the number of requests Z to a particular web server per hour is Poisson(λ). And that the request comes from within the US with probability p.

- Let X be the number of requests per hour from the US and let Y be the number of requests per hour from outside the US. What is the joint pmf of X and Y? Are they independent?

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	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \leq x, s \leq y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds\, dt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_{y} p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy$

Independence (continuous random variables)

Definition. Let X and Y be continuous random variables. The **joint pdf** of X and Y is

$$f_{X,Y}(a, b) \neq \Pr(X = a, Y = b)$$

Definition. The **joint range** of $p_{X,Y}$ is

$$\Omega(X, Y) = \{(c, d) : p_{X,Y}(c, d) > 0\} \subseteq \Omega(X) \times \Omega(Y)$$
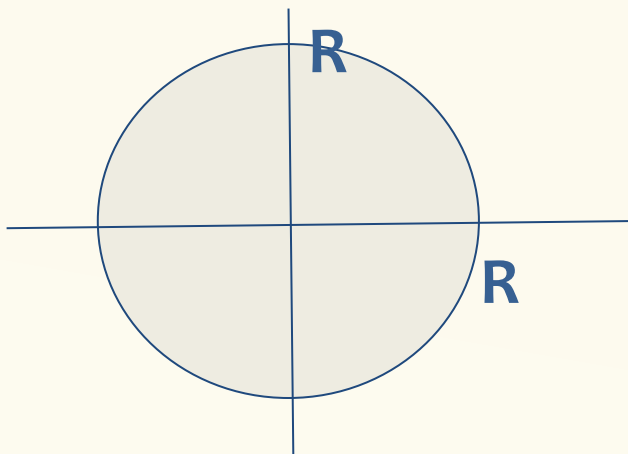
Definition. X and Y are **independent** iff for all a, b

$$f_{X,Y}(a, b) = f_X(a) \cdot f_Y(b)$$

- Suppose that the surface of a disk is a circle with area R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
  - What is their joint density f(x,y)?
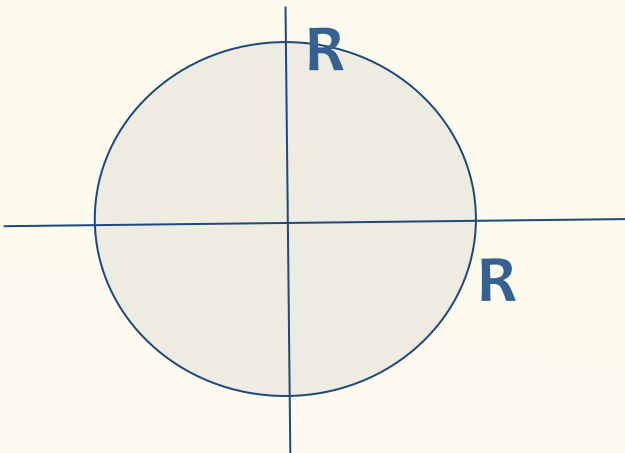
R

R

- Suppose that the surface of a disk is a circle with area R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
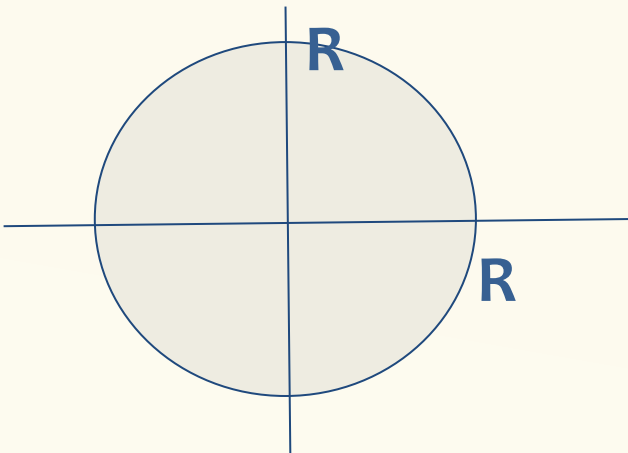 - What is the range of X & Y and the marginal density of X and of Y?

R

R

- Suppose that the surface of a disk is a circle with area R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
 - Are X and Y independent?

Poll:
Are X and Y independent?
a. yes
b. no

- Suppose that the surface of a disk is a circle with area R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
  - What is E(Z)?
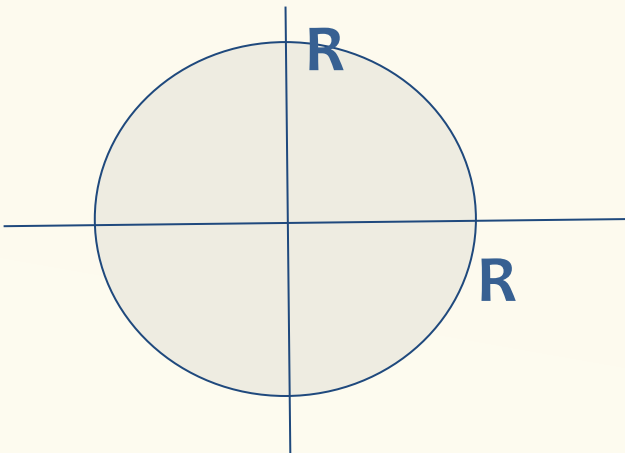
All of this generalizes to more than 2 random variables

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \leq x, s \leq y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds\, dt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$