

CSE 312

Foundations of Computing II

Lecture 17: The Normal Distribution



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Slide Credit: Based on Stefano Tessaro's slides for 312 19au incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & myself 😊

Agenda

- Law of Total Expectation (LTE) Practice
- Chebyshev's Inequality
- The Normal Distribution
- Practice with Normals



Example: Flipping Coins

$$\text{LTE: } E[X] = \sum_{i=1}^{\infty} E[X|A_i] P(A_i)$$

$$Y = 2$$

Suppose wanted to analyze flipping a random number of coins. Suppose someone gave us $Y \sim \text{Poi}(5)$ fair coins and we wanted to compute the expected number of heads X from flipping those coins.


$$E[X] = 0.5 \cdot ?$$

$$E[X] = \sum_{\omega \in \Omega_Y} \frac{E[X|Y=y]}{P(Y=y)}$$

$$E[X|Y=2] = 0.5 \cdot 2$$

$$\sum_{y=0}^{\infty} 0.5y \cdot e^{-5} \cdot \frac{5^y}{y!}$$

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Using variance

- If we know more about the random variable, e.g. its variance, we can get a better bound!

Chebyshev's Inequality

Markov's inequality

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

Theorem. Let X be a random variable. Then, for any $t > 0$,

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq t) \leq \frac{\text{Var}(X)}{t^2}.$$

Proof: Define $Z = X - \mathbb{E}(X)$

$$\mathbb{E}[(X - \mathbb{E}[X])^2]$$

Definition of Variance

$$\mathbb{P}(|Z| \geq t) = \mathbb{P}(Z^2 \geq t^2) \leq \frac{\mathbb{E}(Z^2)}{t^2} = \frac{\text{Var}(X)}{t^2}$$

$|Z| \geq t$ iff $Z^2 \geq t^2$

Markov's inequality ($Z^2 \geq 0$)

Example – Binomial Random Variable

Chebychev's Inequality

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq t) \leq \frac{\text{Var}(X)}{t^2}.$$

Let X be Binomial RV with parameters. $n, p = 0.5$

$$\mathbb{E}(X) = \frac{n}{2}$$

$$\text{Var}(X) =$$

$$np(1-p) = \frac{n}{4} - \frac{n}{4}$$

What is the probability that $X \geq \frac{3n}{4}$?

Chebychev's inequality: $\mathbb{P}\left(X \geq \frac{3n}{4}\right) \leq \mathbb{P}\left(X - \frac{n}{2} \geq \frac{n}{4}\right)$

$$\mathbb{P}\left(|X - \frac{n}{2}| \geq \frac{n}{4}\right)$$

Markov's inequality: $\mathbb{P}\left(X \geq \frac{3n}{4}\right) \leq \frac{4}{3n} \cdot \frac{n}{2} = \frac{2}{3}$


Tail Bounds

Useful for approximations of complex systems. How good the approximation is depends on the actual distribution and the context you are using it in.

- Usually loose upper-bounds are okay when designing for worst-case

Generally, the more you know about your random variable the better tail bounds you can get.

Agenda

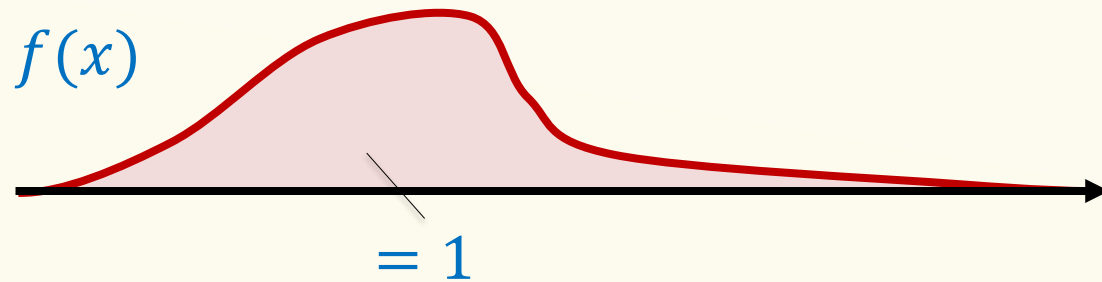
- Law of Total Expectation (LTE) Practice
- Chebyshev's Inequality
- **The Normal Distribution** 
- Practice with Normals

Review – Continuous RVs

Probability Density Function (PDF).

$f: \mathbb{R} \rightarrow \mathbb{R}$ s.t.

- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x) dx = 1$

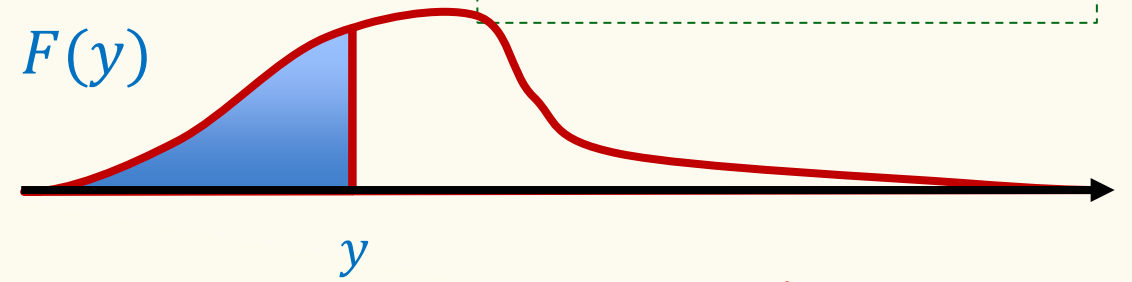


Density \neq Probability !

Cumulative Density Function (CDF).

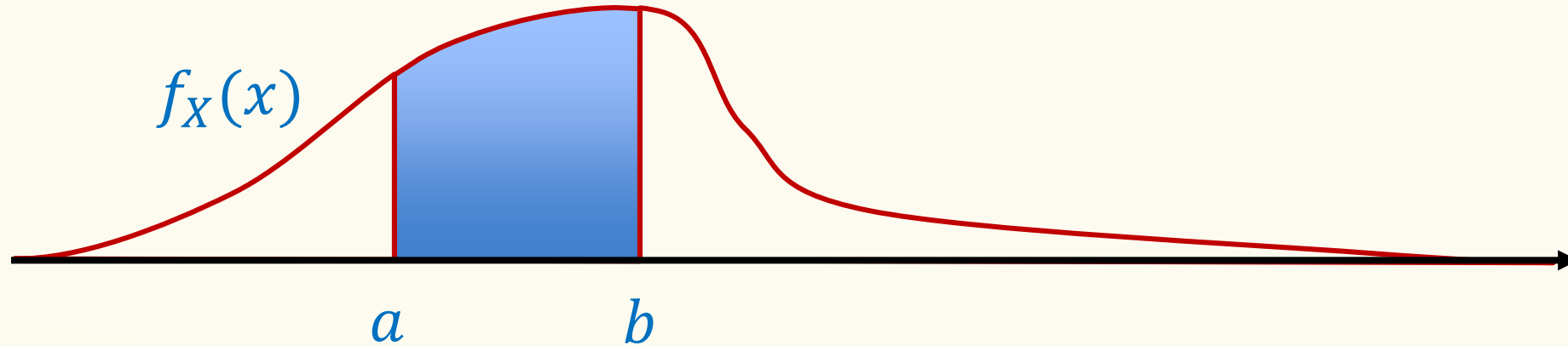
$$F(y) = \int_{-\infty}^y f(x) dx$$

Theorem. $f(x) = \frac{dF(x)}{dx}$



$$\mathbb{P}(a \leq X \leq b) = F_X(b) - F_X(a)$$
$$F(y) = \mathbb{P}(X \leq y)$$

Review – Continuous RVs



$$\mathbb{P}(X \in [a, b]) = \int_a^b f_X(x) dx = F_X(b) - F_X(a)$$

Exponential Distribution

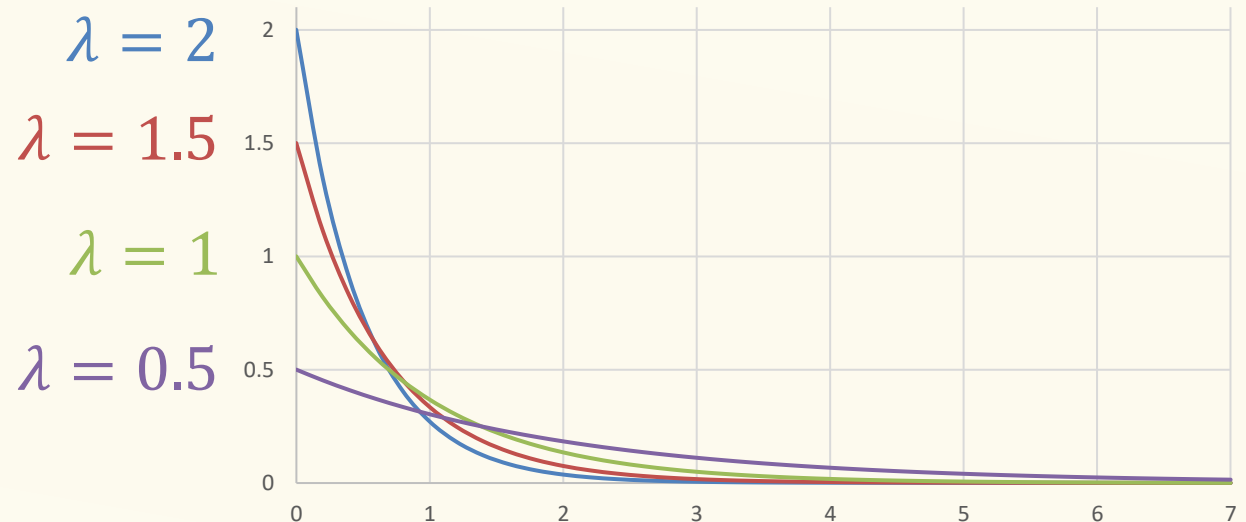
Definition. An **exponential random variable** X with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write $X \sim \text{Exp}(\lambda)$ and say X that follows the exponential distribution.

CDF: For $y \geq 0$,

$$F_X(y) = 1 - e^{-\lambda y}$$

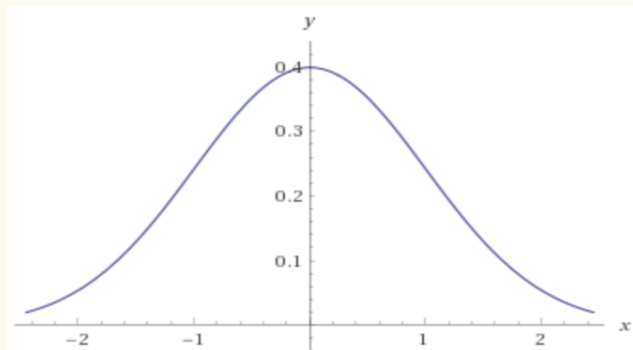


The Normal Distribution

Definition. A **Gaussian (or normal) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$)



Carl Friedrich
Gauss

The Normal Distribution



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Gauss

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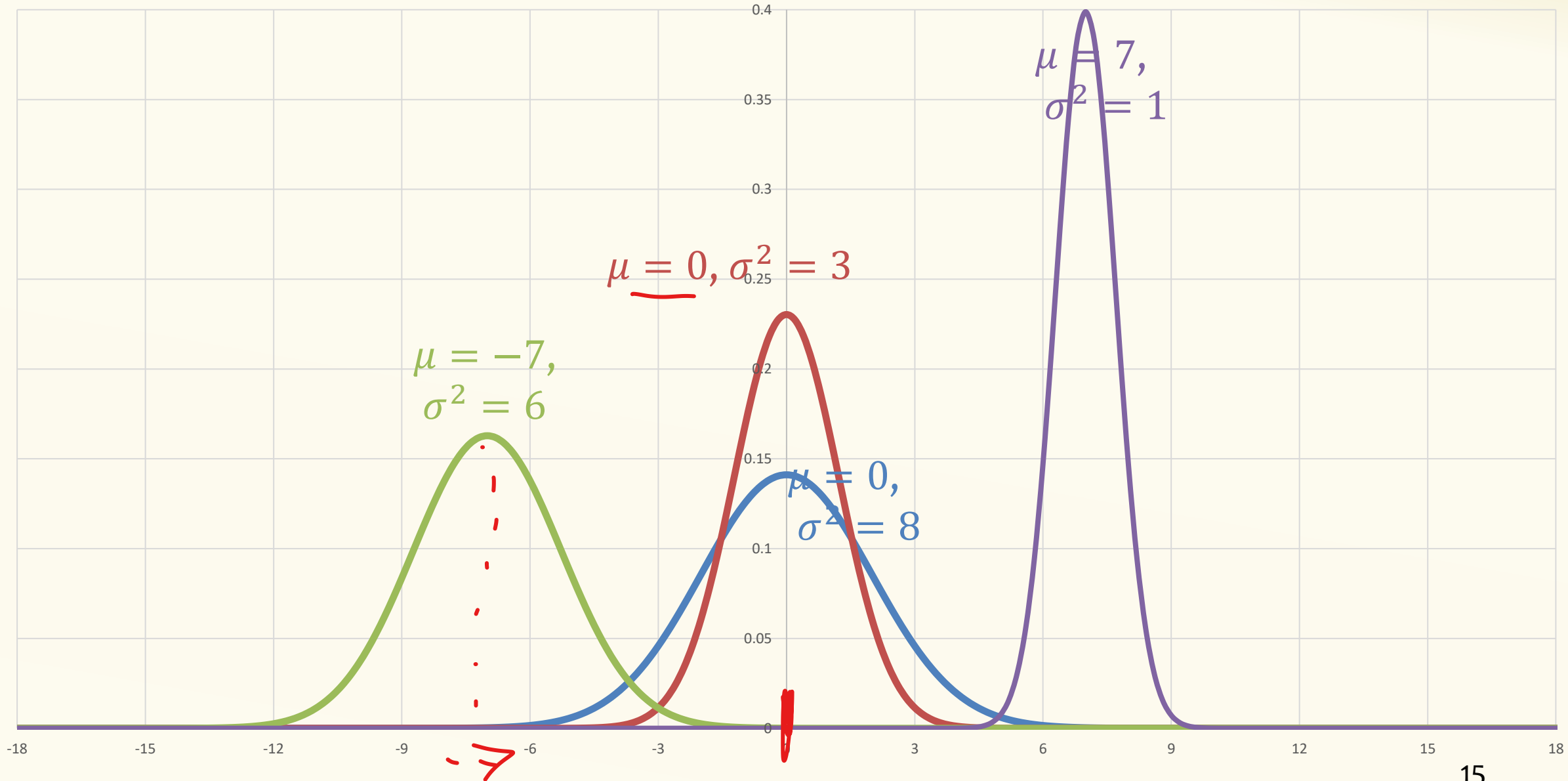
(We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$)

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}(X) = \mu$, and $\text{Var}(X) = \sigma^2$

Expectation follows from density being symmetric around μ , $f_X(\mu - x) = f_X(\mu + x)$

The Normal Distribution

Aka a “Bell Curve” (imprecise name)



Shifting and Scaling – turning one normal dist into another

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Proof. $\mathbb{E}(Y) = a \mathbb{E}(X) + b = a\mu + b$

$$\text{Var}(Y) = a^2 \text{Var}(X) = a^2\sigma^2$$

$X \cdot \frac{1}{\sigma}$
 $\text{Var}(X) = \frac{1}{\sigma^2}$

Can show with algebra that the PDF of $Y = aX + b$ is still normal.

Note: $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$

CDF of normal distribution

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Standard (unit) normal $Z \sim \mathcal{N}(0, 1)$

CDF. $\Phi(z) = \mathbb{P}(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-x^2/2} dx$ for $Z \sim \mathcal{N}(0, 1)$

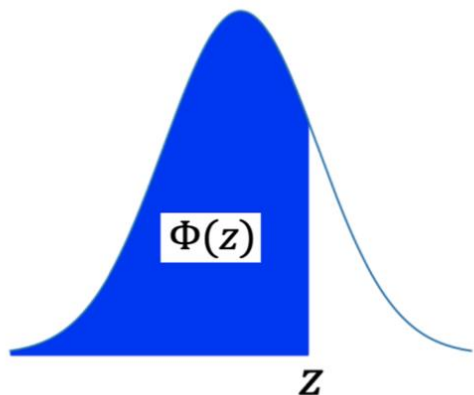
Note: $\Phi(z)$ has no closed form – generally given via tables

$$\Phi(z) = \mathbb{P}(Z \leq z)$$

Table of $\Phi(z)$ CDF of Standard Normal Distn

Make sure to use the one linked on the site!

$$\begin{aligned}
 P(Z \leq 0.42) &= \Phi(0.42) \\
 &= 0.66276
 \end{aligned}$$



Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0, 1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Example

Let $X \sim \mathcal{N}(0.4, 4)$.

$$\mu = 0.4$$

$$\sigma^2 = 4$$

$$\frac{X - \mu}{\sigma}$$

$$\mathbb{P}(X \leq 1.2) = \mathbb{P}\left(\frac{X - 0.4}{2} \leq \frac{1.2 - 0.4}{2}\right)$$

$$= \mathbb{P}(Z \leq 0.4)$$

$$= \Phi(0.4) = \boxed{0.65542}$$

Example

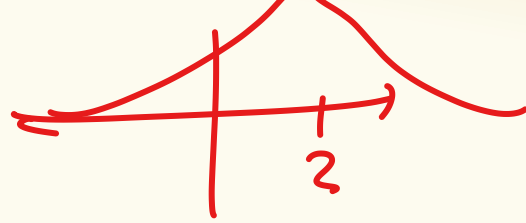
Let $X \sim \mathcal{N}(0.4, 4 = 2^2)$.

$$\begin{aligned}\mathbb{P}(X \leq 1.2) &= \mathbb{P}\left(\frac{X - 0.4}{2} \leq \frac{1.2 - 0.4}{2}\right) \\ &= \mathbb{P}\left(\frac{X - 0.4}{2} \leq 0.4\right) = \Phi(0.4) \approx 0.6554\end{aligned}$$

$\sim \mathcal{N}(0, 1)$

0.1	0.5398	0.5438
0.2	0.5793	0.5832
0.3	0.6179	0.6217
0.4	0.6554	0.6591
0.5	0.6915	0.6950
0.6	0.7257	0.7291
0.7	0.7580	0.7611

Example



Let $X \sim \mathcal{N}(3, 16)$.

$$\mathbb{P}(2 < X < 5) = \mathbb{P}\left(-\frac{1}{4} \leq \frac{X-3}{4} \leq \frac{1}{2}\right)$$

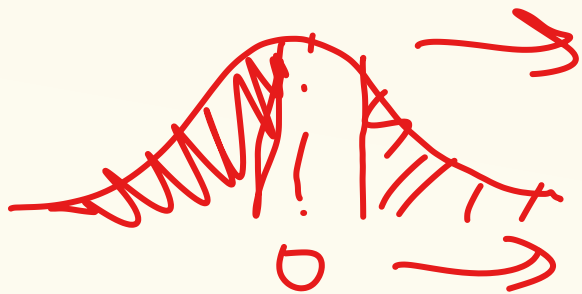
$$\frac{5-3}{\sqrt{16}}$$

$$\frac{2-3}{4}$$

$$\mathbb{P}(Z \geq 0.25) = \mathbb{P}\left(-\frac{1}{4} \leq Z \leq \frac{1}{2}\right) = \mathbb{P}(Z \leq -0.25)$$

$$1 - \mathbb{P}(Z \leq 0.25) = \Phi(0.5) - \Phi(-0.25)$$

$$\Phi(0.5) - (1 - \Phi(0.25))$$



Example

Let $X \sim \mathcal{N}(3, 16)$.

$$\begin{aligned}\mathbb{P}(2 < X < 5) &= \mathbb{P}\left(\frac{2 - 3}{4} < \frac{X - 3}{4} < \frac{5 - 3}{4}\right) \\ &= \mathbb{P}\left(-\frac{1}{4} < Z < \frac{1}{2}\right) \\ &= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{4}\right) \\ &= \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{4}\right)\right) \approx 0.29017\end{aligned}$$

Example – Off by Standard Deviations

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

$$\mathbb{P}(|X - \mu| < k\sigma) =$$

Example – Off by Standard Deviations

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

$$\begin{aligned}\mathbb{P}(|X - \mu| < k\sigma) &= \mathbb{P}\left(\frac{|X - \mu|}{\sigma} < k\right) = \\ &= \mathbb{P}\left(-k < \frac{X - \mu}{\sigma} < k\right) = \Phi(k) - \Phi(-k)\end{aligned}$$

e.g. $k = 1$: 68%, $k = 2$: 95%, $k = 3$: 99%

Summary of procedure for doing calculations with normal r.v.

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$

Therefore,

$$F_X(z) = \mathbb{P}(X \leq z) = \mathbb{P}\left(\frac{X - \mu}{\sigma} \leq \frac{z - \mu}{\sigma}\right) = \Phi\left(\frac{z - \mu}{\sigma}\right)$$

CDF of normal distribution

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Standard (unit) normal $Z \sim \mathcal{N}(0, 1)$

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Note: $\Phi(z)$ has no closed form – generally given via tables

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F_X(z) = \mathbb{P}(X \leq z) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \leq \frac{z-\mu}{\sigma}\right) = \Phi\left(\frac{z-\mu}{\sigma}\right)$

Closure of the normal -- under addition



Fact. If $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ (both independent normal RV) then $aX + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$

Note: The special thing is that the sum of normal **RVs is still a normal RV.**

The values of the expectation and variance is not surprising.

- Linearity of expectation (always true)
- When X and Y are independent, $Var(aX + bY) = a^2Var(X) + b^2Var(Y)$