Chapter 0. Prerequisites

0.3: Sum and Product Notation

Slides (Google Drive) | Alex Tsun | Video (YouTube)

0.3.1 Summation Notation

Suppose that we want to write the sum: $1+2+3+5+6+7+8+9+10$. We can write out each element, but it becomes tedious. We could use dots, to signify this as: $1+2+\cdots+9+10$, but this can become vague if the pattern isn't as clear. Instead, we can use summation notation as shorthand for summations of values. Here we are referring to the sum of each element i, where i will take on every value in the range starting with 1 and ending with 10.

$$1+2+3+\cdots+10=\sum_{i=1}^{10} i$$

Note that i is just a dummy variable. We could have also used j, k, or any other letter. What if we wanted to sum numbers that weren't consecutive integers?

As long as there is some pattern, we can write it compactly! For example, how could we write $16+25+36+\cdots+81$? In the first equation below (0.3.1), j takes on the values from 4 to 9, and the square of each of these values will be summed together. Note that this is equivalent to k taking on the values of 1 to 6, and adding 3 to each of the values before squaring and summing them up (0.3.2).

$$16+25+36+\cdots+81=\sum_{j=4}^{9} j^2 \tag{0.3.1}$$

$$=\sum_{k=1}^{6}(k+3)^2 \tag{0.3.2}$$

If you know what a for-loop is (from computer science), this is exactly the following (in Java or C++). This first loop represents the first sum with dummy variable j.

```
int sum = 0
for (int j = 4; j <= 9; j++) {
    sum += (j * j)
}
```

This second loop represents the second sum with dummy variable k, and is equivalent to the first.

```
int sum = 0
for (int k = 1; k <= 6; k++) {
    sum += ((k + 3) * (k + 3))
}
```

This brings us to the following definition of summation notation:

Definition 0.3.1: Summation Notation

Let $x_1, x_2, x_3, \ldots$ be a sequence of numbers. Then, the following notation represents the "sub-sum":

$$x_a + x_{a+1} + \cdots + x_{b-1} + x_b = \sum_{i=a}^{b} x_i$$

Furthermore, if S is a set, and $f : S \to \mathbb{R}$ is a function defined on S, then the following notation sums over all elements $x \in S$ of $f(x)$:

$$\sum_{x \in S} f(x)$$

Note that the sum over no terms (the empty set) is defined as 0.

Example(s)

Write out the following sums:

- $\sum_{k=3}^{7} k^{10}$
- $\sum_{y \in S} (2^y + 5)$, for $S = \{3, 6, 8, 11\}$
- $\sum_{t=6}^{8} 4$
- $\sum_{z=2}^{1} \sin(z)$
- $\sum_{x \in T} 13x$, for $T = \{-1, -3, 5\}$.

Solution

- For, $\sum_{k=3}^{7} k^{10}$, we raise each value of k from 3 to 7 to the power of 10 and sum them together. That is:

$$\sum_{k=3}^{7} k^{10} = 3^{10} + 4^{10} + 5^{10} + 6^{10} + 7^{10}$$

- Then, if we let $S = \{3, 6, 8, 11\}$, for $\sum_{y \in S} (2^y + 5)$, raise 2 to the power of each value y in S and add 5, and then sum the results together. That is

$$\sum_{y \in S} (2^y + 5) = (2^3 + 5) + (2^6 + 5) + (2^8 + 5) + (2^{11} + 5)$$

- For the sum of a constant, $\sum_{t=6}^{8} 4$, we add the constant, 4 for each value $t = 6, 7, 8$. This is equivalent to just adding 4 together three times.

$$\sum_{t=6}^{8} 4 = 4 + 4 + 4$$

- Then, for a range with no values, the sum is defined as 0, for $\sum_{z=2}^{1} \sin(z)$, because there are no values from 2 to 1, we have:

$$\sum_{z=2}^{1} \sin(z) = 0$$

- Finally, if we let $T = \{-1, -3, 5\}$, for $\sum_{x \in T} 13x$, we multiply each value of x in T by 13 and then sum them up.

$$\begin{aligned}\sum_{x \in T} 13x &= 13(-1) + 13(-3) + 13(5) \\ &= 13(-1 + -3 + 5) \\ &= 13 \sum_{x \in T} x\end{aligned}$$

Notice that we can actually factor out the 13; that is, we could sum all values of $x \in T$ first, and then multiply by 13. This is one of a few properties of summations we can see below!

□

Further, the associative and distributive properties hold for sums. If you squint hard enough, you can kind of see why they're true! We'll also see some examples below too, since the notation can be confusing at first.

Fact 0.3.1: The Associative and Distributive Properties of Sums

We have the associative property (0.3.3) and distributive property (0.3.4, 0.3.5) for sums.

$$\sum_{x \in A} f(x) + \sum_{x \in A} g(x) = \sum_{x \in A} (f(x) + g(x)) \tag{0.3.3}$$

$$\sum_{x \in A} \alpha f(x) = \alpha \sum_{x \in A} f(x) \tag{0.3.4}$$

$$\left(\sum_{x \in A} f(x)\right)\left(\sum_{y \in B} g(x)\right) = \sum_{x \in A} \sum_{y \in b} f(x)g(y) \tag{0.3.5}$$

The last property is like FOIL - if you multiply $(x + x^2 + x^3)(1/y + 1/y^2)$ (left-hand side) for example, you would have to sum over every possible combination $x/y + x/y^2 + x^2/y + x^2/y^2 + x^3/y + x^3/y^2$ (right-hand side).

The proof of these are left to the reader, but see the examples below for some intuition!

Example(s)

"Prove" the following by writing out the sums:

- $\sum_{i=5}^{7} i + \sum_{i=5}^{7} i^2 = \sum_{i=5}^{7} (i + i^2)$
- $\sum_{j=3}^{5} 2j = 2 \sum_{j=3}^{5} j$
- $(\sum_{i=1}^{2} f(a_i))(\sum_{j=1}^{3} g(b_j)) = \sum_{i=1}^{2} \sum_{j=1}^{3} f(a_i)g(b_j)$

Solution

- Looking at the associative property, we know the following:

$$\sum_{i=5}^{7} i + \sum_{i=5}^{7} i^2 = (5+6+7) + (5^2+6^2+7^2) = (5+5^2) + (6+6^2) + (7+7^2) = \sum_{i=5}^{6} (i+i^2)$$

- Also, using the distributive property we know:

$$\sum_{j=3}^{5} 2j = 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5 = 2(3+4+5) = 2\sum_{j=3}^{5} j$$

- This one is similar to FOIL. Finally, we have:

$$\begin{aligned}
\left(\sum_{i=1}^{2} f(a_i)\right)\left(\sum_{j=1}^{3} g(b_j)\right) &= (f(a_1)+f(a_2))(g(b_1)+g(b_2)+g(b_3)) \\
&= f(a_1)g(b_1) + f(a_1)g(b_2) + f(a_1)g(b_3) + f(a_2)g(b_1) + f(a_2)g(b_2) + f(a_2)g(b_3) \\
&= \sum_{i=1}^{2}\sum_{j=1}^{3} f(a_i)g(b_j)
\end{aligned}$$

$\square$

0.3.2 Product Notation

Similarly, we can define product notation to handle multiplications.

Definition 0.3.2: Product Notation

Let $x_1, x_2, x_3, \ldots$ be a sequence of numbers. Then, the following notation represents the "sub-product" $x_a \cdot x_{a+1} \cdot \cdots \cdot x_{b-1} \cdot x_b$:

$$\prod_{i=a}^{b} x_i$$

Further, if S is a set, and $f : S \to \mathbb{R}$ is a function defined on S, then the following notation multiplies over all elements $x \in S$ of $f(x)$:

$$\prod_{x \in S} f(x)$$

Note that the product over no terms is defined as 1 (not 0 like it was for sums).

Example(s)

Write out the following products:

- $\prod_{a=4}^{7} a$
- $\prod_{x \in S} 8$ for $S = \{3, 6, 8, 11\}$

- $\prod_{z=2}^{1} \sin(z)$
- $\prod_{b=2}^{5} 9^{1/b}$

Solution

- For $\prod_{a=4}^{7} a$, we multiply each value a in the range 4 to 7 and have:

$$\prod_{a=4}^{7} a = 4 \cdot 5 \cdot 6 \cdot 7$$

- Then if, we let $S = \{3, 6, 8, 11\}$, for $\prod_{x \in S} 8$, we multiply 8 for each value in the set, S and have:

$$\prod_{x \in S} 8 = 8 \cdot 8 \cdot 8 \cdot 8$$

- Then for $\prod_{z=2}^{1} \sin(z)$, we have the empty product, because there are no values in the range 2 to 1, so we have:

$$\prod_{z=2}^{1} \sin(z) = 1$$

- Finally for $\prod_{b=2}^{5} 9^{1/b}$, we have each value of b from 2 to 5 of $9^{1/b}$, to get

$$\begin{aligned}\prod_{b=2}^{5} 9^{1/b} &= 9^{1/2} \cdot 9^{1/3} \cdot 9^{1/4} \cdot 9^{1/5} \\ &= 9^{1/2+1/3+1/4+1/5} \\ &= 9^{\sum_{b=2}^{5} 1/b}\end{aligned}$$

$\square$

Also, if you were to do the same examples as we did for sums replacing $\prod$ with $\sum$, you just multiply instead of add! They are almost identical, except the empty sum is 0 and the empty product is 1.