CSE 312: Foundations of Computing II
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Lecture Topics: 4.4 Transforming Continuous RVs

[**Tags:** Transforming Continuous RVs]

1. Suppose $X \sim Exp\left(\lambda = \frac{1}{2}\right)$ is the waiting time in hours until your pizza delivery arrives, and suppose we decide to tip $Y = g(X) = \frac{24}{X+1}$ dollars.
   a. What is the range, PDF, and CDF of X? Hint: You can look this up.
   b. What is the range Ω_Y?
   c. Find $F_Y(y)$ using the CDF method, then find $f_Y(y)$ afterwards.
   d. Find $f_Y(y)$ using the explicit formula, after verifying the monotonicity and invertibility criteria.
   e. Set up integrals for $E[Y]$ in two ways: one with LOTUS and $f_X(x)$, and one with $f_Y(y)$. Explicitly define your limits of integration and the integrand so that one could enter your integral into WolframAlpha.

Solution:

a. Since $X \sim Exp\left(\lambda = \frac{1}{2}\right)$, we have

$$\Omega_X = [0, \infty)$$
$$f_X(x) = \frac{e^{-x/2}}{2} \text{ if } x \in \Omega_X$$
$$F_X(x) = 1 - e^{-x/2} \text{ if } x \in \Omega_X$$

b. When $X = 0$ (the lowest value), we tip $Y = 24$ dollars, and if $X \to \infty$, we tip $Y \to 0$ dollars. So the range is $\Omega_Y = (0,24)$.

c. Let $y \in \Omega_Y$. We have

$$F_Y(y) = P(Y \leq y) \text{ [def of CDF]}$$
$$= P\left(\frac{24}{X+1} \leq y\right) \text{ [def of } Y\text{]}$$
$$= P(24 \leq Xy + y)$$
$$= P\left(X \geq \frac{24-y}{y}\right)$$
$$= 1 - P\left(X < \frac{24-y}{y}\right)$$
$$= 1 - P\left(X \leq \frac{24-y}{y}\right) \text{ [since P(X = k) = 0]}$$
$$= 1 - F_X\left(\frac{24-y}{y}\right)$$
$$= 1 - \left(1 - e^{-\frac{24-y}{2y}}\right)$$
$$= e^{-\frac{24-y}{2y}}$$

$$f_Y(y) = \frac{d}{dy} e^{-\frac{24-y}{2y}} = \frac{12\, e^{-\frac{24-y}{2y}}}{y^2}, \qquad y \in \Omega_Y$$

d. We know $\frac{1}{X+1}$ is a monotone decreasing function, so $\frac{24}{X+1}$ is as well. The inverse of g is $h(y) = \frac{24-y}{y} = \frac{24}{y} - 1$. We solved for this inverse as follows:

$$g(x) = \frac{24}{x+1} = y$$
$$y(x+1) = 24$$
$$x + 1 = \frac{24}{y}$$
$$x = \frac{24}{y} - 1 = h(y)$$

We can compute $h'(y) = -\frac{24}{y^2}$.

Then, the formula says

$$f_Y(y) = f_X(h(y))|h'(y)| = \frac{1}{2} e^{-\frac{1}{2}\left(\frac{24-y}{y}\right)} \cdot \left|-\frac{24}{y^2}\right| = \frac{12\, e^{-\frac{24-y}{2y}}}{y^2}$$

This matches the other way as well!

e. The two ways:

By LOTUS:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx = \int_0^{\infty} \left(\frac{24}{x+1}\right)\left(\frac{1}{2} e^{-\frac{1}{2}x}\right) dx$$

By definition of expectation:

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^{24} y \left(\frac{12\, e^{-\frac{24-y}{2y}}}{y^2}\right) dy$$

[Bonus!]

2. Suppose $X \sim Unif(-1,1)$ (continuous), then find the PDF of $Y = X^2$. (Hint: Use the CDF method; why does the explicit formula not work in this case?).

Since $X \sim Unif(-1,1)$ we know that its CDF of X is:

$$F_X(x) = \frac{x+1}{2}$$

and the range of X is:

$$\Omega_Y = [-1,1]$$

so the range of $Y = X^2$ is:

$$\Omega_Y = [0,1]$$

Then, we can find the CDF of Y by doing the following:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P\left(-\sqrt{y} \leq X \leq \sqrt{y}\right) \\ &= F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right) \\ &= \frac{\sqrt{y}+1}{2} - \frac{-\sqrt{y}+1}{2} \\ &= \sqrt{y} \end{aligned}$$

We can differentiate the CDF to get the PDF. That is:

$$f_Y(y) = \frac{d}{dy}\sqrt{y}$$

$$= \frac{1}{2\sqrt{y}}$$

So, for the final PDF we have:

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 \leq y \leq 1 \\ 0 & otherwise \end{cases}$$