CSE 312: Foundations of Computing II
Instructor: Alex Tsun
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Lecture Topics: 5.2 Joint Continuous Distributions

[**Tags:** Joint PDFs, Marginal PDFs, Expectation]

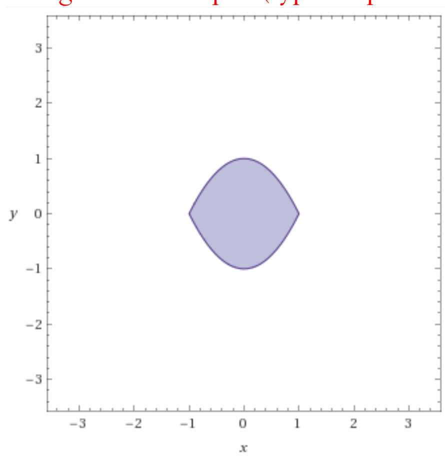
1. Suppose we have a joint density as follows:

$$f_{X,Y}(x,y) = \begin{cases} cx^2y^4, & x^2 - 1 \le y \le 1 - x^2 \\ 0, & \text{otherwise} \end{cases}$$

   a. Sketch the joint range $\Omega_{X,Y}$, and determine the marginal ranges Ω_X, Ω_Y from the picture.
   b. Write an expression for the value c that makes $f_{X,Y}$ a valid joint density function.
   c. Write an expression for $f_X(x)$.
   d. Write an expression for $f_Y(y)$.
   e. Write an expression for $E\left[\frac{1}{X^2+Y^2}\right]$.

Solution:

a. Using WolframAlpha (type in "plot x^2 – 1 <= y and y <= 1-x^2"),



We can see that $\Omega_X = \Omega_Y = [-1,+1]$.

b. We need to set up a double integral to equal 1, and here $dydx$ is going to be easier than $dxdy$ (this would require two integrals).

For each value of $x \in [-1,1]$, we draw the vertical line and see that we go from $y = x^2 - 1$ to $y = 1 - x^2$. Just rearrange and solve for c:

$$\int_{-1}^{1}\int_{x^2-1}^{1-x^2} cx^2y^4dydx = 1 \rightarrow c = \frac{1}{\int_{-1}^{1}\int_{x^2-1}^{1-x^2} x^2y^4dydx}$$

If you did $dxdy$, we would have different cases for when $y \in [-1,0]$ vs $y \in [0,1]$. If $y \in [-1,0]$, we pick a point on the y-axis in this range, and draw a horizontal line. The bottom curve is $y = x^2 - 1$. Where does x enter and leave? It enters at $x = -\sqrt{1+y}$ and leaves at $x = \sqrt{1+y}$. Similarly, if $y \in [0,1]$, we have from $x = -\sqrt{1-y}$ to $x = \sqrt{1-y}$. Hence, the total integral is

$$\int_{-1}^{0}\int_{-\sqrt{1+y}}^{\sqrt{1+y}} cx^2y^4dxdy + \int_{0}^{1}\int_{-\sqrt{1-y}}^{\sqrt{1-y}} cx^2y^4dxdy = 1$$
$$\rightarrow c = \frac{1}{\int_{-1}^{0}\int_{-\sqrt{1+y}}^{\sqrt{1+y}} x^2y^4dxdy + \int_{0}^{1}\int_{-\sqrt{1-y}}^{\sqrt{1-y}} x^2y^4dxdy}$$

c. We did the work earlier:

$\Omega_X = [-1,1]$ and for $x \in \Omega_X$:

$$f_X(x) = \int_{x^2-1}^{1-x^2} cx^2y^4dy$$

d. The range is $\Omega_Y = [-1,1]$. There are two cases like in part (b):

If $y \in [-1,0]$, then

$$f_Y(y) = \int_{-\sqrt{1+y}}^{\sqrt{1+y}} cx^2y^4dx$$

And if $y = [0,1]$, then

$$f_Y(y) = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} cx^2y^4dx$$

e. By LOTUS,

$$E\left[\frac{1}{X^2+Y^2}\right] = \int_{-1}^{1}\int_{x^2-1}^{1-x^2} \frac{1}{x^2+y^2} cx^2y^4dydx$$

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2. Suppose (X, Y, Z) are jointly distributed with density function:

$$f_{X,Y,Z}(x, y, z) = \begin{cases} ce^{-13x}e^{-13y}, & x, y > 0 \text{ and } 0 < z < 47 \\ 0, & \text{otherwise} \end{cases}$$

The joint range is a nice "rectangle" (rectangular prism with infinitely large base technically), which we unfortunately can't visualize.

a. Set up an appropriate triple integral with the order $dxdydz$ for the value of c, which turns out to be $13^2/47$.
b. Compute $f_X(x)$ using WolframAlpha after setting up the limits of integration (you should integrate over ALL other variables). And note $f_Y(y)$ is identical by symmetry. Actually, X, Y have the same distribution from our zoo – which is it?
c. Compute $f_Z(z)$ using WolframAlpha. Actually, Z also has a distribution from our zoo – which is it?
d. Are X, Y, Z mutually independent? (This means, $f_{X,Y,Z}(x, y, z) = f_X(x)f_Y(y)f_Z(z)$, and this also must hold true for each pair's marginal joint distribution, but let's not worry about that now.) Note that the joint range must also be the Cartesian product of the three marginal ranges.
e. Write an expression for $E\left[\log\left(\frac{1}{Z^{X+Y}}\right)\right]$.

__Solution__:

a. We must have

$$\int_0^{47}\int_0^{\infty}\int_0^{\infty} ce^{-13x}e^{-13y}dxdydz = 1 \to c = \frac{1}{\int_0^{47}\int_0^{\infty}\int_0^{\infty} e^{-13x}e^{-13y}dxdydz} = \frac{13^2}{47}$$

b. We have $\Omega_X = (0, \infty)$ and for $x \in \Omega_X$:

$$f_X(x) = \int_0^{47}\int_0^{\infty} ce^{-13x}e^{-13y}dydz = 13e^{-13x}$$

Hence $X \sim Exp(13)$. (And same for Y).

c. We have $\Omega_Z = (0,47)$ and for $z \in \Omega_X$

$$f_Z(z) = \int_0^{\infty}\int_0^{\infty} ce^{-13x}e^{-13y}dxdy = \frac{1}{47}$$

Hence, $Z \sim Unif(0,47)$ (the continuous uniform).

d. We first check that the range is a "rectangle" (rectangular prism with infinitely large base), this indicates that there is a chance for independence. Then, we check the following:

$$f_X(x)f_Y(y)f_Z(z) = 13e^{-13x} \cdot 13e^{-13y} \cdot \frac{1}{47} = \frac{13^2}{47} e^{-13x} e^{-13y} = f_{X,Y,Z}(x, y, z)$$

and it turns out they are independent!

e. By LOTUS,

$$E\left[\log\left(\frac{1}{Z^{X+Y}}\right)\right] = \int_0^{47} \int_0^{\infty} \int_0^{\infty} \log\left(\frac{1}{z^{x+y}}\right) \cdot ce^{-13x} e^{-13y} dxdydz$$