

[Tags: Linearity of Expectation, Indicator Random Variables]

1. There is a team of 14 DJs at Rainy Dawg Radio that are getting ready for a show and each need to choose an unordered playlist of 3 **different** songs from 50 total songs. Each DJ chooses a playlist of 3; any subset of 3 songs out of 50 is equally likely. DJ's choose songs independently of each other. That is, it is possible to reuse the same song across different DJs, but a DJ cannot use a song more than once.
 - a. The manager of the station needs to buy the rights to the songs the DJs choose for any show. How many songs should she expect to buy the rights to for the show?
 - b. The rights to each song cost \$100, and there is an overall processing fee of \$40 to complete all the transactions. What is the expected cost to the manager of the station?

Solution:

a.

Let X be the number of songs for which we need to buy the rights for the show. For $i = 1, \dots, 50$, let $X_i = 1$ if there's at least one DJ that chooses song i and $X_i = 0$ otherwise.

Then by independence, $E[X_i] = P(X_i = 1) = 1 - P(X_i = 0)$.

$$P(X_i = 0) = P(\text{no one chose song } i) = P(\text{one DJ did not choose song } i)^{14}$$

$$\left(\frac{\binom{49}{3}}{\binom{50}{3}} \right)^{14}$$

$$\text{So } E[X_i] = 1 - P(X_i = 0) = 1 - \left(\frac{\binom{49}{3}}{\binom{50}{3}} \right)^{14}.$$

So, by linearity of expectation we have:

$$\begin{aligned} E[X] &= E[X_1] + E[X_2] + \dots + E[X_{50}] \\ &= 50E[X_1] \\ &= 50 \left(1 - \left(\frac{\binom{49}{3}}{\binom{50}{3}} \right)^{14} \right) \end{aligned}$$

b.

The cost is $100X + 40$, so linearity of expectation we have:

$$E[100X + 40] = 100E[X] + 40$$

where $E[X]$ is our answer from part (a).

[Tags: Expectation, Variance, LOTUS]

2. Suppose we have two coins. Coin C_1 comes up heads with probability 0.3 and coin C_2 comes up heads with probability 0.9. We repeat this process 3 times:
- Choose a coin with equal probability.
 - Flip that coin once.

Suppose X is the number of heads after 3 flips.

- What is $E[X]$?
- What is $Var(X)$?
- Based on the number of heads we get, we earn $Y = \frac{1}{X+1}$ dollars. What is $E[Y]$?

Solution:

- We can use linearity of expectation or find the PMF directly.

Let X be the number of heads. At each trial,

$$\begin{aligned} P(H) &= P(H|C_1)P(C_1) + P(H|C_2)P(C_2) \text{ [LTP]} \\ &= 0.3 \cdot 0.5 + 0.9 \cdot 0.5 = 0.6 \end{aligned}$$

If we want to use linearity, let X_i for $i = 1, 2, 3$ be 1 if the i^{th} flip was heads and 0 otherwise. Then,

$$E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = P(X_i = 1) = 0.6 \text{ [above]}$$

$$E[X] = E[X_1 + X_2 + X_3] = E[X_1] + E[X_2] + E[X_3] = 1.8 \text{ [LoE]}$$

Otherwise, we have $\Omega_X = \{0, 1, 2, 3\}$.

$$p_X(k) = \begin{cases} 0.4^3, & k = 0 \\ \binom{3}{1} 0.6^2 \cdot 0.4, & k = 1 \\ \binom{3}{2} 0.6 \cdot 0.4^2, & k = 2 \\ 0.6^3, & k = 3 \end{cases}$$

$$E[X] = \sum_k k \cdot p_X(k) = 0 \cdot 0.4^3 + \dots + 3 \cdot 0.6^3 = 1.8$$

- To find variance, we need $E[X^2]$, for which we will use LOTUS:

$$E[X^2] = \sum_k k^2 \cdot p_X(k) = 0^2 \cdot 0.4^3 + \dots + 3^2 \cdot 0.6^3 = 3.96$$

$$Var(X) = E[X^2] - E[X]^2 = 3.96 - 1.8^2 = 0.72$$

- We use LOTUS:

$$\begin{aligned} E\left[\frac{1}{X+1}\right] &= \sum_k \frac{1}{k+1} p_X(k) \\ &= \frac{1}{0+1} 0.4^3 + \frac{1}{1+1} \binom{3}{1} 0.6^2 \cdot 0.4 + \frac{1}{2+1} \binom{3}{2} 0.6 \cdot 0.4^2 + \frac{1}{3+1} 0.6^3 \\ &= 0.406 \end{aligned}$$