CSE 312: Foundations of Computing II
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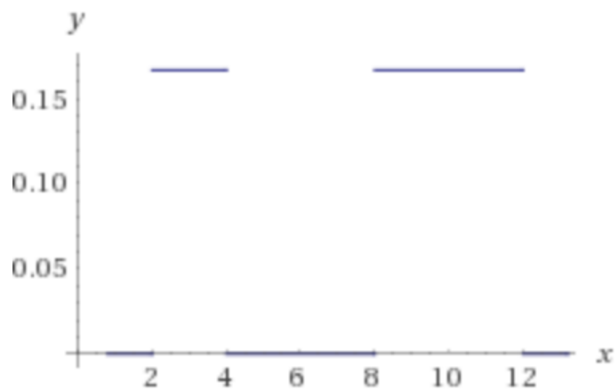
Lecture Topics: 4.1 Continuous Random Variable Basics

[**Tags:** PDFs, CDFs]

1. Alex decided he wanted to create a "new" type of distribution that will be famous, but he needs some help. He knows he wants it to be continuous and have uniform density, but he needs help working out some of the details. We'll denote a random variable X having the "Uniform-2" distribution as $X \sim Unif2(a, b, c, d)$, where $a < b < c < d$. We want the density to be non-zero in $[a, b]$ and $[c, d]$, and zero everywhere else. Anywhere the density is non-zero, it must be equal to the same constant.

Here's an example of $Unif2(2, 4, 8, 12)$:



a. Find the PDF $f_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty, \infty)$. Use a piecewise function.
b. Find the CDF $F_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty, \infty)$. Use a piecewise function.

Solution:

a. We need to just find the height h which is constant. The area under the curve is two rectangles, and we need it to be 1, so

$$(b-a)h + (d-c)h = 1 \to h = \frac{1}{(b-a)+(d-c)}$$

Hence,

$$f_X(x) = \begin{cases} \frac{1}{(b-a)+(d-c)}, & x \in [a,b] \cup [c,d] \\ 0, & \text{otherwise} \end{cases}$$

b. The CDF is the cumulative area to the left of a certain point. We have 5 cases.

If $x < a$, the probability $X \leq x$ is 0, since there's no area to the left or no probability of this happening.
If $x \geq d$, the probability $X \leq x$ is 1, since it is guaranteed that $X \leq d$ (the total area to the left is 1).
If $b \leq x < c$, the probability $X \leq x$ is just the area of the left retangle, which is base * height or

$$(b-a) \cdot \frac{1}{(b-a)+(d-c)}.$$

If $a \leq x < b$, then we have a smaller subrectangle of base $x - a$, multiplied by height, which is

$$(x-a) \cdot \frac{1}{(b-a)+(d-c)}$$

If $c \leq x < d$, we have the entire left rectangle, and a smaller subrectangle of base $x - c$, which is

$$(x-c) \cdot \frac{1}{(b-a)+(d-c)} + (b-a) \cdot \frac{1}{(b-a)+(d-c)}$$

So, we have:

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{(b-a)+(d-c)}, & a \leq x < b \\ \frac{b-a}{(b-a)+(d-c)}, & b \leq x < c \\ \frac{(x-c)+(b-a)}{(b-a)+(d-c)}, & c \leq x < d \\ 1, & x \geq d \end{cases}$$

[**Tags:** PSet3 Q2, PDFs, CDFs, Expectation]

2. A flea of negligible size is trapped in a large, spherical, inflated beach ball with radius r. At this moment, it is equally likely to be at any point within the ball. Let X be the distance of the flea from the center of the ball.
    a. Find the range of X, Ω_X.
    b. Find the cumulative distribution function $F_X(x) = P(X \leq x)$. To get full-credit, make sure you define it for all real numbers, by possibly using a piecewise function. (Hint: Interpret it in English first, and the volume of a sphere with radius t is $\frac{4}{3}\pi t^3$).
    c. Find the probability density function $f_X(x)$. To get full-credit, make sure you define it for all real numbers, by possibly using a piecewise function.
    d. Find an integral for $E[X]$.

Solution: Watch lecture ☺