CSE 312: Foundations of Computing II Autumn 2023

Problem Set 2

Due: Thursday, October 12, by 11:59am

Instructions

Solutions format. Every step in your solution should be explained carefully. The logical reasoning behind your solution should be sound and evident from your write-up.

For example, if you are asked to compute the number of ways to permute the set $\{1, 2, 3, 4\}$ that start with 1 or 2, it is not enough to provide the answer 12. A complete approach would explain that (1) we can count separately the permutations starting with 1 and those starting with 2, and that (2) the two sets are disjoint, and hence the overall number is the sum of the numbers of permutations of each type. Then, (3) explain that there are $3!$ permutations of each type. Finally, (4) say that the overall number totals to $2 \cdot 3! = 12$.

A higher number of mathematical symbols in your solution will not make your solution more precise or "better" – what *is* important is that the logical flow is complete and can be followed by the graders. Relying exclusively on mathematical symbols can in fact often make the solution less readable. Avoid expressions such as "it easy to see" and "clearly" – just explain these steps.

Also, you may find the following short note (by Francis E. Su at Harvey Mudd) helpful.

Unless a problem states otherwise, you can leave your answer in terms of factorials, combinations, etc., for instance 26^7 or $26!/7!$ or $26 \cdot \binom{26}{7}$ are all good forms for final answers.

Collaboration policy. You are required to submit your own solutions. You are allowed to discuss the homework with other students. However, the write up must clearly be your own, and moreover, you must be able to explain your solution at any time. We reserve ourselves the right to ask you to explain your work at any time in the course of this class.

Late policy. You have a total of **six** late days during the quarter, but can only use up to three late days on any one problem set. Please plan ahead, as we will not be willing to add any additional late days except in absolute, verifiable emergencies. The final problem set will not be accepted late (however, it will be due only on Friday of the last week of class).

Solutions submission. You must submit your solution via Gradescope. In particular:

- Submit a *single* PDF file containing the solution to all tasks in the homework. Each numbered task should be solved on its own page (or pages). Follow the prompt on Gradescope to link tasks to your pages.
- Do not write your name on the individual pages – Gradescope will handle that.
- We encourage you to typeset your solution. The homepage provides links to resources to help you doing so using LaTeX. If you do use another tool (e.g., Microsoft Word), we request that you use a proper equation editor to display math (MS Word has one). For example, you should be able to write $\sum_{i=1}^{n} x^i$ instead of x^1 + x^2 + ... + x^n. You can also provide a handwritten solution, as long as it is on a single PDF file that satisfies the above submission format requirements. It is your responsibility to make sure handwritten solutions are readable – we will *not* grade unreadable write-ups.

Task 1 – Thinking Combinatorially [15 pts]

We saw in Lecture 3 that combinatorial proofs can be more elegant than algebraic proofs and also provide insights into an equation that goes beyond algebra. In this task, our goal is to develop the skill and intuition for such proofs.

a) At the library you can choose 2 books from m fiction books and n non-fiction books. Express the total number of options in terms of the number of the number of fiction-only options, the number of non-fiction-only options and the number of mixed options. Use your counts of the numbers for each of these options to derive a formula relating $\binom{n+m}{2}$, $\binom{m}{2}$, $\binom{n}{2}$ and m, and n.

b) Prove combinatorially that $\sum_{j=0}^{k} \binom{n}{j} \cdot \binom{m}{k-j} = \binom{n+m}{k}$.

c) Use the formula from part **b)** to show that $\sum_{j=0}^{n} \left(\binom{n}{j}\right)^2 = \binom{2n}{n}$.

Task 2 – Pigeons in Pigeonholes [10 pts]

All pairs of people in a group compare the differences measured in numbers of days between the day of the month of their birthdays to yield their "birthday difference". (For example, if one is born on January 7 and the other is born on May 4, their birthday difference would be 3.)

Prove that in any group of 12 people there will be at least three pairs of people who have the same birthday difference. (these pairs may overlap).

Task 3 – The Encoding Method [12 pts]

In this task, you should use the encoding/(stars and bars) technique of counting possibilities or use one of the formulas we have previously derived using that technique.

a) You need to guess a five-digit number (with digits from 0 to 9), and the only clue you are given is that the digits in the number are non-increasing. For example, 99831 and 00000 are valid numbers, while 21346 and 57889 are not. How many valid numbers are there?

b) How many solutions do we have for the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 12$, assuming all x_i are non-negative integers and x_1 is odd?

Task 4 – Binomial Theorem [12 pts]

a) What is the coefficient of x^4y^9 in the expansion of $(3x - 2y^3)^7$?

b) Use the binomial theorem to prove that for every positive integer n,

$$\sum_{i=0}^{2n} \binom{2n}{i} (-4)^{2n-i} = 3^{2n}$$

Task 5 – Principle of Inclusion and Exclusion [15 pts]

There are three couples sitting at a round table with six seats. We think of two seating assignments as the same if one can be rotated to give the other one (for example, the seating arrangement ABCDEF is the same as BCDEFA starting at the same seat). How many ways can the three couples be seated at the table if ...

a) ... there are no restrictions?

b) ... no one sits next to their partner (on either side)?

Task 6 – Balls in Jar [21 pts]

A jar contains 7 purple, 8 gold, and 10 white balls. Three times in a row, we pick a random ball (uniformly) from the jar, and then discard it (i.e., we do not add it back to the jar). This is referred to as sampling *without* replacement.

a) What is the probability that all balls have different colors? Give your answer as a simplified fraction.

b) What is the probability that all balls have the same color? Give your answer as a simplified fraction.

c) What is the probability that the first ball is purple and the two remaining balls are gold?

d) Re-compute the above probabilities for parts **a)** through **c)** for the case of sampling *with* replacement, i.e., the ball is re-inserted into the jar after having been chosen.

Task 7 – Random Questions [15 pts]

a) What is the probability that the digit 6 doesn't appear among n digits where each digit is one of (0-9) and all sequences are equally likely?

b) Suppose you randomly permute the numbers $1, 2, \ldots, n$, (where $n > 100$). That is, you select a permutation uniformly at random. What is the probability that the number 25 ends up in the 25-th position in the resulting permutation? (For example, in the permutation $1, 3, 2, 5, 4$ of the numbers $1 \ldots 5$, the number 2 is in the 3rd position in the permutation and the number 4 is in the 5th position.)

c) A fair coin is flipped n times (each outcome in $\{H, T\}^n$ is equally likely). What is the probability that all heads occur at the end of the sequence? (The case that there are no heads is a special case of having all heads at the end of the sequence, i.e. 0 heads.)