CSE 312: Foundations of Computing II Autumn 2023

Problem Set 7

Due: Wednesday, November 22, by 11:59pm

Instructions

Solutions format, collaboration policy, and late policy. See PSet 1 for further details. The same requirements and policies still apply. Also follow the typesetting instructions from the prior PSets.

Solutions submission. You must submit your solution via Gradescope. In particular:

- Submit a PDF file containing your answers to task 1-4 to "Pset7 [written]" on Gradescrope. Submit another PDF file containing your answer to task 5 (the extra credit), if you choose to complete it, to "Pset 7 [Extra credit]". Each numbered task should be solved on its own page (or pages). Follow the prompt on Gradescope to link tasks to your pages.
- Do not write your name on the individual pages – Gradescope will handle that.
- We encourage you to typeset your solution. The homepage provides links to resources to help you doing so using LaTeX. If you do use another tool (e.g., Microsoft Word), we request that you use a proper equation editor to display math (MS Word has one). For example, you should be able to write $\sum_{i=1}^{n} x^i$ instead of x^1 + x^2 + ... + x^n. You can also provide a handwritten solution, as long as it is on a single PDF file that satisfies the above submission format requirements. It is your responsibility to make sure handwritten solutions are readable – we will *not* grade unreadable write-ups.

Task 1 – Take me to a Higher Level [25 pts]

Shreya is playing League of Legends[1] for H hours, where H is a random variable, equally likely to be 1, 2 or 3. The level L that she gets to is random and depends on how long she plays for. We are told that

$$\mathbb{P}(L = \ell \mid H = h) = \frac{1}{h}, \qquad \text{for } \ell = 1, \ldots, h\,.$$

a) (5 points) Find the joint distribution of L and H.

b) (5 points) Find the marginal distribution of L.

c) (5 points) Find the conditional distribution of H given that $L = 1$ (that is, $\mathbb{P}(H = h \mid L = 1)$ for each possible h in 1,2,3). Use the definition of conditional probability and the results from previous parts.

d) (10 points) Suppose that we are told that Shreya got to level 1 or 2. Find the expected number of hours she played conditioned on this event, defined (by definition of conditional expectation) as follows:

$$\mathbb{E}\left[H \mid L = 1 \cup L = 2\right] = \sum_{h=1}^{3} h \cdot \mathbb{P}(H = h \mid L = 1 \cup L = 2)$$

[1] Whoever wrote this problem has never played League of Legends so if it doesn't make sense in the context of the actual game, just pretend it does.

Task 2 – Bug Fixing [15 pts]

A software development team of 15 developers is working on a large codebase. They have a bug tracking system and the number of bugs that they find follows a Poisson random variable with mean 6. Each developer chooses one of the bugs to work on uniformly at random and independent of each other's choices (these developers are not great at communicating and may end up working on the same task). The developers simultaneously work on fixing the bugs. Each developer independently fixes their chosen bug with probability 0.8, use the law of total expectation to compute the expected number of bugs that are fixed (a bug is fixed if at least one developer fixes it).
Hint: Think about partitioning on cases depending on the number of bugs that are found.

Task 3 – Frosting Requires Concentration [20 pts]

Chef Bob is slowly decorating 150 cupcakes for a big party. It takes an average of 3 minutes for Chef Bob to frost and decorate each cupcake, with a standard deviation of 0.5 minutes. The time to frost and decorate each cupcake is independent.

a) What is the expectation of the total time to frost and decorate all the cupcakes?

b) What is the variance of the total time to frost and decorate all the cupcakes?

c) Bob will have 10 hours (600 minutes) to frost and decorate all of the cupcakes between now and when his customer is coming to pick them up. Use Markov's Inequality to give a *lower bound* on the probability that Bob finishes the cupcakes before the customer comes to pick them up.

d) Can we improve the lower bound from **c)** using Chebyshev's inequality? If so, what bound do you get?

Task 4 – Chebyshev and Pairs of Events [20 pts]

Chebyshev's inequality can be used to bound the tails of random variables that are sums of random variables that are not independent, provided that pairs of random variables are not too positively correlated with each other. Let random variable $X = \sum_{i=1}^{n} X_i$ be the sum of indicator variables $X_1, \ldots, X_n$ for events $A_1, \ldots, A_n$. Suppose that $\mathbb{E}[X_i] = \mathbb{P}(A_i) = p$ for each i and that for every $i \neq j$, we have $\mathbb{E}[X_i X_j] = \mathbb{P}[A_i \cap A_j] \leqslant p^2$.

a) Use linearity of expectation to show that $\mathbb{E}[(X_1 + X_2)^2] \leqslant 2p + 2p^2$.

b) Use linearity of expectation and the fact that $(\sum_{i=1}^{n} X_i)^2 = \sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n} \sum_{j \neq i} X_i X_j$ to show that $\mathbb{E}[X^2]$ is at most $np + n(n-1)p^2$. Then compute an upper bound on $\text{Var}(X)$.

c) Use Chebyshev's inequality to show that $\mathbb{P}(X \leqslant np/4) \leqslant \frac{16(1-p)}{9np}$.

Task 5 – Extra Credit: I just want something to happen

Sometimes we have a non-negative integer-valued random variable Y that is at most n and is not always 0 and we want to get a lower bound on the probability that it is bigger than 0, $\mathbb{P}(Y \geqslant 1) = 1 - \mathbb{P}(Y = 0)$. (This often is of interest when Y corresponds to the number of events that happen from some collection of n possibilities.)

a) What does Chebyshev's inequality give as a lower bound for this probability in terms of $\mathbb{E}[Y]$ and $\mathbb{E}[Y^2]$? (Simplify your answer as much as possible.)

b) In order for the lower bound above to be non-trivial (bigger than 0), how much larger can $\mathbb{E}[Y^2]$ be than $\mathbb{E}[Y]^2$?

c) Prove the following bound, which is non-trivial for all values of $\mathbb{E}[Y^2]$ and $\mathbb{E}[Y]$?

$$\mathbb{P}(Y \geqslant 1) \geqslant \frac{\mathbb{E}[Y]^2}{\mathbb{E}[Y^2]}$$

Hint: Try to bound $\mathbb{E}[Y^2] \cdot \mathbb{P}(Y \geqslant 1)$. The Cauchy-Schwarz inequality is helpful here. It states that for real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ we have

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geqslant \left(\sum_{i=1}^{n} a_i b_i\right)^2.$$