Section 2

Review

1) **Subsets/Binomial Coefficients** The number of ways to choose a k-element subset of a set of n elements is _____.

2) **Binomial theorem.** $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}$: $(x+y)^n =$ ________________________.

3) **Inclusion-exclusion.** $|A \cup B| =$ ________________________.

4) **Inclusion-exclusion.** $|A \cup B \cup C| =$ ____.

5) **Pigeonhole principle.** If there are n pigeons and k holes, and $n > k$, some hole has at least ________________________ pigeons.

6) **Multinomial coefficients.** Suppose there are n objects, but only k are distinct, with $k \leqslant n$. (For example, "godoggy" has $n = 7$ objects (characters) but only $k = 4$ are distinct: (g, o, d, y)). Let n_i be the number of times object i appears, for $i \in \{1, 2, \ldots, k\}$. (For example, $(3, 2, 1, 1)$, continuing the "godoggy" example.) The number of distinct ways to arrange the n objects is: ____

7) **Binary encoding(a.k.a Stars and Bars).** The number of ways to distribute n indistinguishable balls into k distinguishable bins is ____.

8) **Probability space.** We call the set of possible outcomes of an experiment the **sample space**(denoted Ω), $\omega \in \Omega$ an **outcome** of the experiment, $E \subseteq \Omega$ an **event**, and $\mathbb{P} : \Omega \to [0, 1]$ the **probability measure/function**. A **probability space** is a pair $(\Omega, \mathbb{P})$ where we have $\mathbb{P}(\omega)$ ________ for all $\omega \in \Omega$ and $\sum_{\omega \in \Omega} \mathbb{P}(\omega) =$ ________.

9) **Mutually exclusive events.** The events $\mathcal{A}$ and $\mathcal{B}$ are *mutually exclusive* if $\mathcal{A} \cap \mathcal{B} =$ _____________

10) **Additivity of Probability.** If $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are mutually exclusive events, then $\mathbb{P}\left(\bigcup_{i=1}^n \mathcal{A}_i\right) =$ ___________________ .

11) **Complement.** For any event $\mathcal{A}$, $\mathbb{P}(\mathcal{A}^c) =$ ___________________ .

12) **Uniform Probability Space.** A probability space $(\Omega, \mathbb{P})$ where $\mathbb{P}(\omega) =$ ___________________ for all $\omega \in \Omega$

13) **Equally Likely Outcomes.** If every outcome in a finite sample space Ω is equally likely(i.e. we have an uniform prob. space), and E is an event, then $\mathbb{P}(E) =$ ___________________ .

Task 1 – HBCDEFGA

How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

Task 2 – Ingredients

Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other. For example, "INGREEDINT" is invalid because the two E's are adjacent.

Task 3 – Card Party

At a card party, someone brings out a deck of bridge cards (4 suits with 13 cards in each). N people each pick 2 cards from the deck and hold onto them. What is the minimum value of N that guarantees at least 2 people have the same combination of suits?

Task 4 – The Pigeonhole Principle

Show that in any group of n people there are two who have an identical number of friends within the group. (Friendship is bi-directional – i.e., if A is friend of B, then B is friend of A – and nobody is a friend of themselves.)

Solve in particular the following two cases individually:

a) Everyone has at least one friend.

b) At least one person has no friends.

Task 5 – A Team and a Captain

Give a combinatorial proof of the following identity:

$$n\binom{n-1}{r-1} = \binom{n}{r}r.$$

Hint: Consider two ways to choose a team of size r out of a set of size n and a captain of the team (who is also one of the team members).

Task 6 – Balls from an Urn

Suppose that an urn (a fancy name for a jar that doesn't have a lid) contains one red ball, one blue ball, and one green ball. (Other than their colors, balls are identical.) Imagine we draw two balls *with replacement*, i.e., after drawing one ball we put it back into the urn before we draw the second one. (In particular, each ball is equally likely to be drawn.)

a) Give a probability space describing the experiment.

b) What is the probability that both balls are red? (Describe the event first, before you compute its probability.)

c) What is the probability that at most one ball is red?

d) What is the probability that we get at least one green ball?

e) Repeat **c)-d)** for the case where the balls are drawn *without replacement*, i.e., when the first ball is drawn, it is not placed back from the urn.

Task 7 – Shuffling Cards

We have a deck of cards, with 4 suits, and 13 cards in each suit. Within each suit, the cards are ordered Ace $>$ King $>$ Queen $>$ Jack $>$ 10 $> \cdots >$ 2. Also, suppose we perfectly shuffle the deck (i.e., all possible shuffles are equally likely).

What is the probability the first card on the deck is (strictly) larger than the second one?

Task 8 – Robot Wears Socks

Suppose Joe is a k-legged robot, who wears a sock and a shoe on each leg. Suppose he puts on k socks and k shoes in some order, each equally likely. Each action is specified by saying whether he puts on a sock or a shoe, and saying which leg he puts it on. In how many ways can he put on his socks and shoes in a valid order? We say an ordering is valid if, for every leg, the sock gets put on before the shoe. Assume all socks are indistinguishable from each other, and all shoes are indistinguishable from each other.

Task 9 – Congressional Tea

Twenty politicians are having tea, 6 Democrats and 14 Republicans.

a) If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans? (We assume every possible way of giving tea is equally likely.)

b) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 8 Republicans and 2 Democrats? (We assume every possible way of giving tea is equally likely.)

Task 10 – Trick or Treat

Suppose on Halloween, someone is too lazy to keep answering the door, and leaves a jar of exactly N total candies. You count that there are exactly K of them which are kit kats (and the rest are not). The sign says to please take exactly n candies. Each item is equally likely to be drawn. Let X be the number of kit kats we draw (out of n). What is $\mathbb{P}(X = k)$, that is, the probability we draw exactly k kit kats?

Task 11 – Binomial Theorem

What is the coefficient of z^{36} in $(-2x^2yz^3 + 5uv)^{312}$?