Problem Set 2

Due: Wednesday, January 18, by 11:59pm

Instructions

Solutions format. Every step in your solution should be explained carefully. The logical reasoning behind your solution should be sound and evident from your write-up.

For example, if you are asked to compute the number of ways to permute the set $\{1, 2, 3, 4\}$ that start with 1 or 2, it is not enough to provide the answer 12. A complete approach would explain that (1) we can count separately the permutations starting with 1 and those starting with 2, and that (2) the two sets are disjoint, and hence the overall number is the sum of the numbers of permutations of each type. Then, (3) explain that there are $3!$ permutations of each type. Finally, (4) say that the overall number totals to $2 \cdot 3! = 12$.

A higher number of mathematical symbols in your solution will not make your solution more precise or "better" – what *is* important is that the logical flow is complete and can be followed by the graders. Relying exclusively on mathematical symbols can in fact often make the solution less readable. Avoid expressions such as "it easy to see" and "clearly" – just explain these steps.

Also, you may find the following short note (by Francis E. Su at Harvey Mudd) helpful.

Unless a problem states otherwise, you can leave your answer in terms of factorials, combinations, etc., for instance 26^7 or $26!/7!$ or $26 \cdot \binom{26}{7}$ are all good forms for final answers.

Collaboration policy. The written problems on this pset may be done with a **single partner**. In this case, only one person will submit the written part on Gradescope and add their partner as a collaborator. Task 7 (coding) must be done on your own and will be submitted separately.

Individuals and pairs are still encouraged to discuss problem-solving strategies with other classmates as well as the course staff, but each pair must write up their own solutions and, as stated above, submit a single joint homework.

Late policy. You have a total of **six** late days during the quarter, but can only use up to two late days on any one problem set. Please plan ahead, as we will not be willing to add any additional late days except in absolute, verifiable emergencies.

Solutions submission. You must submit your solution via Gradescope. In particular:

- Submit a *single* PDF containing the solution to all of Tasks 1-6 to Gradescope under "**PSet 2 [Written]**". Task 7 has a coding and written portion, so under "**Pset 2 [Coding]**" you will be uploading a pdf of your written solutions to task 7 (including the plot we ask you to make) and a .py file called cse312_pset2_pingpong.py.

- Each numbered task should be solved on its own page (or pages). Follow the prompt on Gradescope to link tasks to your pages.

- Do not write your name on the individual pages – Gradescope will handle that.
- We encourage you to typeset your solution. The homepage provides links to resources to help you doing so using LaTeX. If you do use another tool (e.g., Microsoft Word), we request that you use a proper equation editor to display math (MS Word has one). For example, you should be able to write $\sum_{i=1}^{n} x^i$ instead of $\mathrm{x\hat{}1 + x\hat{}2 + \ldots + x\hat{}n}$. You can also provide a handwritten solution, as long as it is on a single PDF file that satisfies the above submission format requirements. It is your responsibility to make sure handwritten solutions are readable – we will *not* grade unreadable write-ups.

Task 1 – Thinking Combinatorially [15 pts]

We saw in Lecture 3 that combinatorial proofs can be more elegant than algebraic proofs and also provide insights into an equation that goes beyond algebra. In this task, our goal is to develop the skill and intuition for such proofs. To this end, prove each of the following identities using a *combinatorial argument*; an algebraic solution will be marked substantially incorrect. (Note that $\binom{a}{b}$ is 0 if $b > a$.)

a) (8 points)

$$\sum_{k=0}^{\infty} \binom{m}{k}\binom{n}{k} = \binom{m+n}{n}.$$

Hint: Start with the right hand side and imagine you are choosing a team of n people from a group of people consisting of m Americans and n Canadians.

b) (7 points)

$$\sum_{k=0}^{\infty} \binom{n}{k}\binom{k}{m} = \binom{n}{m}2^{n-m}.$$

Hint: Think about choosing a committee from n people, of which m are leaders.

Task 2 – Pigeons in Pigeonholes [10 pts]

At a dinner party, all of the n people present are to be seated at a circular table. Suppose there is a nametag at each place at the table and suppose that nobody sits down in their correct place. Use the pigeon-hole principle to show that it is possible to rotate the table so that at least two people are sitting in the correct place. Be sure to specify precisely what the pigeons are, precisely what the pigeonholes are, and precisely what the mapping of pigeons to pigeonholes is.

Task 3 – Stars and bars [12 pts]

a) We have 12 people and 36 rooms. How many different ways are there to assign the (distinguishable) people to the (distinguishable) rooms? (Any number of people can go into any of the 36 rooms.)

b) We have 30 identical (indistinguishable) apples. How many different ways are there to place the apples into 20 (distinguishable) boxes? (Any number of apples can go into any of the boxes.)

c) We have 20 identical (indistinguishable) apples. How many different ways are there to place the apples into 6 (distinguishable) boxes, if each box is required to have at least two apples in it.?

Task 4 – Principle of Inclusion and Exclusion [10 pts]

How many positive integers are there less than 1000 that are relatively prime to 100, i.e., have no common factor with 100?

Task 5 – Sample Spaces and Probabilities [18 pts]

For each of the following scenarios first describe the sample space and indicate how big it is (i.e., what its cardinality is) and then answer the question, assuming a uniform probability space.

a) You flip a fair coin 100 times. What is the probability of exactly 30 heads?

b) You roll 2 fair 6-sided dice, one red and one blue. What is the probability that the sum of the two values showing is 5?

c) You are given a random 5 card poker hand (selected from a single deck, order doesn't matter). What is the probability you have a full-house (3 cards of one rank and 2 cards of another rank)?

d) 10 labeled balls are placed into 20 labeled bins (with each placement equally likely). What is the probability that bin 1 contains exactly 3 balls?

e) There are 24 psychiatrists and 30 psychologists attending a certain conference. Three of these 54 people are randomly chosen to take part in a panel discussion. What is the probability that at least one psychologist is chosen? What is the probability that exactly three psychologists are chosen?

f) You buy 12 cupcakes choosing from 3 different types (chocolate, vanilla and caramel). Cupcakes of the same type are indistinguishable. What is the probability that you have at least one of each type?

Task 6 – Random Questions [15 pts]

a) What is the probability that the digit 7 doesn't appear among 100 digits where each digit is one of (0-9) and all sequences are equally likely?

b) Suppose you randomly permute the numbers $1, 2, \ldots, 100$. That is, you select a permutation uniformly at random. What is the probability that the number 75 ends up in the 45-th position in the resulting permutation? (For example, in the permutation $1, 3, 2, 5, 4$ of the numbers $1 \ldots 5$, the number 2 is in the 3rd position in the permutation and the number 4 is in the 5th position.)

c) A fair coin is flipped 100 times (each outcome in $\{H, T\}^{100}$ is equally likely). What is the probability that all heads occur at the end of the sequence? (The case that there are no heads is a special case of having all heads at the end of the sequence, i.e. 0 heads.)

Task 7 – Ping Pong [coding + written] [20 pts]

We'll finally answer the long-awaited question: what's the probability you win a ping pong game up to n points, when your probability of winning each point is p (and your friend wins the point with probability $1-p$)? Assume you have to win by (at least) 2; for example, if $n = 21$ and the score is $21 - 20$, the game isn't over yet.
Write your code for the following parts in the provided file: cse312_pset2_pingpong.py.

a) (5 points) Implement the function part_a.

b) (15 points) Implement the function part_b. This function will NOT be autograded but you will still submit it; you should use the space here to generate the plot asked of you below.

(i) Generate a plot similar to the one shown below in Python (without the watermarks). Details on how to construct it are in the starter code. Attach your plot in your written submission for this part. Your plot should:

- contain plot and axis titles,
- have the same shape as the plot below,
- use three different colors,
- use three different line styles,
- and a legend for the three lines.

(ii) Write AT MOST 2-3 sentences identifying the interesting pattern you notice when n gets larger (regarding the steepness of the curve). Try to explain why it makes sense. (Later in the course, we will see why more formally.)

(iii) Each curve you make for different values of n always (approximately) passes through 3 points. Give the three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, and explain why intuitively this happens in AT MOST 2-3 sentences.

Figure 1: Your plot should look something like this.