Problem Set 6

Due: Wednesday, February 22 by 11:59pm

Instructions

Solutions format and late policy. See PSet 1 for further details. The same requirements and policies still apply. Also follow the typesetting instructions from the prior PSets.

Collaboration policy. The written problems on this pset may be done with a **single partner**. In this case, only one person will submit the written part on Gradescope and add their partner as a collaborator.

Solutions submission. You must submit your solution via Gradescope under "PSet 6 [Written]". This will be a *single* PDF file containing the solution to all tasks in the homework. Each numbered task should be solved on its own page (or pages). Follow the prompt on Gradescope to link tasks to your pages. Do not write your name on the individual pages – Gradescope will handle that.

If you are working with a partner, there should be only one submission for both of you.

Task 1 – PDF [15 pts]

For this exercise, give exact answers as simplified fractions. Define function f_X by

$$f_X(x) = \begin{cases} (1-x^3)/2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

a) Show that f_X has the properties required of a probability density function.

b) Compute the expectation of a random variable X with f_X as its PDF.

c) Compute the variance of a random variable X with f_X as its PDF.

Task 2 – How long? [8 pts]

Suppose that the distance from Seattle to Portland is 180 miles. You decide to buy an electric bicycle for the trip. Suppose that electric bikes have speeds that are uniformly distributed between 15 and 30 miles per hour, and you buy a random motorized bike (i.e., its speed is Uniform(15,30)). Let T be the time it takes you to ride from Seattle to Portland. What is $\mathbb{E}[T]$?
Note: Recall that the indefinite integral of $g(x) = x^{-1}$ is equal to $\ln(x) + C$, assuming $x > 0$.

Task 3 – Practice with uniforms [12 pts]

Suppose that $X \sim Unif(0,1)$. Let a, b be constants such that $0 < a < b < 1$.

a) (3 points) What is $\mathbb{P}(X > b | X > a)$?

b) (3 point) For any real number y, what is $\mathbb{P}(X \leqslant y | X > a)$? Be sure you include all cases.

c) (6 points) Let Y be a random variable such that

$$F_Y(y) = \mathbb{P}(X \leqslant y | X > a).$$

What is $f_Y(y)$ and what kind of random variable is Y? (Note that when you specify a distribution from our zoo, you must specify both the name of the distribution and the values of the parameters of that distribution.)

Task 4 – Dart [8 pts]

You throw a dart at a circular target of radius r. Let X be the distance of your dart's hit from the center of the target. Your aim is such that $X \sim \text{Exponential}(2/r)$. (Note that it is possible for the dart to completely miss the target.)

a) As a function of r, determine the value m such that $\Pr(X < m) = \Pr(X > m)$. Then, for $r = 4$, give the value of m to 3 decimal places.

b) What is the probability that you miss the target completely? Give your answer to 3 decimal places.

Task 5 – Flea [16 pts]

A flea of negligible size is trapped in a large, spherical, inflated beach ball with radius r. (Recall that such a ball has volume $\frac{4}{3}\pi r^3$.) At this moment, it is equally likely to be at any point within the ball. Let X be the distance of the flea from the center of the ball. For X, find ...

a) the cumulative distribution function F_X.

b) the probability density function f_X.

c) the expected value $\mathbb{E}[X]$.

d) the variance $\text{Var}(X)$

Task 6 – Kangaroos [10 pts]

The average leap of a kangaroo is 20 feet. However, because of various factors such as strength, wind, etc, the kangaroo doesn't always leap exactly 20 feet. A zoologist tells you that the kangaroo leap is normally distributed with mean 20 and variance 9.

a) What is the probability that the kangaroo leaps more than 25 feet?

b) What is the probability that the kangaroo leaps between 13 and 27 feet?

Task 7 – CLT for stocks [10 pts]

Suppose that the daily price change of a certain stock on the stock market is a random variable with mean 0 and variance σ^2. Thus, if Y_n is the price of the stock on the n-th day, then

$$Y_n = Y_{n-1} + X_n, \quad n \geqslant 1$$

where $X_1, X_2, \ldots$ are independent, identically distributed random variables with mean 0 and variance σ^2. Suppose also that today's stock price is 100 and $\sigma^2 = 16$. Use the Central Limit Theorem to estimate the probability that the stock price will exceed 110 after 10 days.