CSE 312 Section 2 Slides

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Announcements

- Homework 1 due yesterday
- Homework 2 due next Wednesday (1/18) 11:59 pm PST

Review

- Some important denotation and definition on your handout
- Multinomial
- Inclusion-Exclusion: +singles - doubles + triples - quads + …
- New combinatorics concepts
  - Pigeonhole
  - Stars and Bars

Review

- Intro to Probability

(Countable Additivity) If E and F are *mutually exclusive*, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$. This actually holds for any countable (finite or countably infinite) collection of pairwise mutually exclusive events $E_1, E_2, E_3, \ldots$.

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

Corollaries:

1. (Complementation) $\mathbb{P}(E^C) = 1 - \mathbb{P}(E)$.
2. (Monotonicity) If $E \subseteq F$, then $\mathbb{P}(E) \leq \mathbb{P}(F)$.
3. (Inclusion-Exclusion) $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$.

Theorem 2.1.4: Probability in Sample Space with Equally Likely Outcomes

If Ω is a sample space such that each of the unique outcome elements in Ω **are equally likely**, then for any event $E \subseteq \Omega$:

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|}$$

Question 5: “Count the Solutions”

How many nonnegative integer solutions to $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 70$?

Stars and Bars

70 indistinguishable balls into 6 bins: let a_i be the number of balls in bin i.

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Stars and bars

$$\binom{70+6-1}{6-1} = \binom{75}{5}$$

Question 7: “Card Party”

At a card party, someone brings out a deck of bridge cards (4 suits with 13 cards in each). N people each pick 2 cards from the deck and hold onto them. What is the minimum value of N that guarantees at least 2 people have the same combination of suits?

Pigeonhole principle

We want at least 2 people to have the same combination of suits.

Pigeon:

Pigeonhole:

Pigeonhole principle

We want at least 2 people to have the same combination of suits.

Pigeon: N people

Pigeonhole: ? combination of suits

Pigeonhole principle

Same suit: 4 ways

Different suits: 4C2 = 6 ways

Total: 10 combinations of suits

11 people is enough to guarantee

Question 6: “Spades and Hearts”

Given 3 different spades and 3 different hearts, shuffle them. Compute Pr(E), where E is the event that the suits of the shuffled cards are in alternating order.

If Ω is a sample space such that each of the unique outcome elements in Ω **are equally likely**, then for any event $E \subseteq \Omega$:

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|}$$

Computing probability in the case of **equally likely outcomes** reduces to doing two counting problems (counting $|E|$ and $|\Omega|$, where computing $|\Omega|$ is generally easier than computing $|E|$). Just use the techniques from Chapter 1 (Counting) to do this!

-Textbook

Size of sample space: all possible card orderings

6!

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Size of event:

3! ways to order spades, 3! ways to order hearts
either hearts at the front or spades at the front

$$2 * 3!^2$$

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Answer : $\frac{2 * 3!^2}{6!}$

Alternate solution*

Size of sample space: all possible suits orderings

C(6,3); choose 3 out of the 6 spots for spades

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Size of sample space: all possible suits orderings

C(6,3); choose 3 out of the 6 spots for spades

Size of event:

2; either hearts at the front or spades at the front

Alternate solution*

Size of sample space: all possible suits orderings

C(6,3); choose 3 out of the 6 spots for spades

Size of event:

2; either hearts at the front or spades at the front

Answer : $$\frac{2 * 3!^2}{6!}$$

Make sure that all events are equally likely

Question 12: “Trick or Treat”

Suppose on Halloween, someone is too lazy to keep answering the door, and leaves a jar of exactly ***N*** total candies. You count that there are exactly ***K*** of them which are kit kats (and the rest are not). The sign says to please take exactly n candies. Each item is equally likely to be drawn. Let ***X*** be the number of kit kats we draw (out of ***n***). What is ***Pr(X = k)***, that is, the probability we draw exactly ***k*** kit kats?

If Ω is a sample space such that each of the unique outcome elements in Ω **are equally likely**, then for any event $E \subseteq \Omega$:

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|}$$

Computing probability in the case of **equally likely outcomes** reduces to doing two counting problems (counting $|E|$ and $|\Omega|$, where computing $|\Omega|$ is generally more straightforward than computing $|E|$). Just use the techniques from Chapter 1 (Counting) to do this!

-Textbook

$$\Pr(X = k) = \frac{|E|}{|\Omega|}$$

Size of Sample Space: the total number of ways to choose **n** candies out of **N** total.

$$\Pr(X = k) = \frac{|E|}{\binom{N}{n}}$$

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Size of Event: counted in two stages!

$$\Pr(X = k) = \frac{|E|}{\binom{N}{n}}$$

Size of Sample Space: the total number of ways to choose **n** candies out of **N** total.

1. choose **k** out of the **K** kit kats

2. Then choose **n - k** out of the **N - K** other candies

Size of Event: counted in two stages!

$$\Pr(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

Size of Sample Space: the total number of ways to choose **n** candies out of **N** total.

Question 13: “Weighted Die”

Consider a weighted (6-faced) die such that

- Pr(1) = Pr(2),
- Pr(3) = Pr(4) = Pr(5) = Pr(6), and
- Pr(1) = 3Pr(3).

What is the probability that the outcome is [3 or 4]?

- $Pr(1) = Pr(2)$
- $Pr(3) = Pr(4) = Pr(5) = Pr(6)$
- $Pr(1) = 3Pr(3)$

the sum of probabilities for the sample space must equal 1

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the sum of probabilities for the sample space must equal 1

$$Pr(1) + Pr(2) + Pr(3) + Pr(4) + Pr(5) + Pr(6) = 1$$

- $Pr(1) = Pr(2)$
- $Pr(3) = Pr(4) = Pr(5) = Pr(6)$
- $Pr(1) = 3Pr(3)$

Use the given equations to substitute everything into $Pr(3)$:

$$3Pr(3) + 3Pr(3) + Pr(3) + Pr(3) + Pr(3) + Pr(3) = 10Pr(3) = 1$$

- Pr(3) = 0.1

Pr(3) = Pr(4) = 0.1

- $Pr(3) = 0.1$

$Pr(3) = Pr(4) = 0.1$

$Pr(3 \text{ or } 4) = Pr(3) + Pr(4) = 0.2$

The MatPlotLib Library

Plotting A Graph using matplotlib.pyplot

```
x = np.arange(10)
y = x ** 2
z = 5*x + 7
plt.plot(x, y, "b", label="y = x^2", linestyle='-')
```

The x and y coordinates of the data

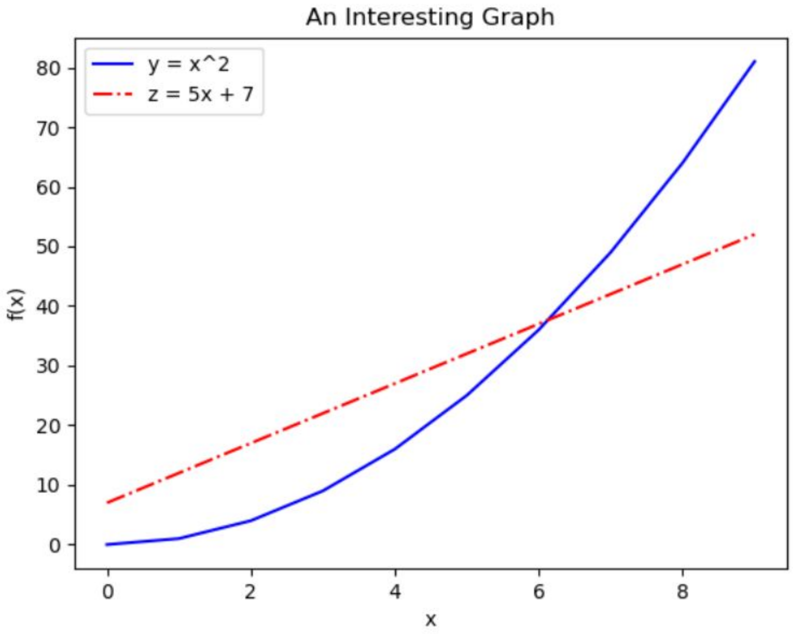
Line color. Some abbreviations available, such as r - red, g - green, b - blue, etc.

Label for line in the legend

Line style. ‘-’ gives a solid line, ‘--’ gives a dashed one, ‘-.’ gives a dash-dot one, etc.

```python
import matplotlib.pyplot as plt
import numpy as np

x = np.arange(10)
y = x ** 2
z = 5*x + 7
plt.plot(x, y, "b", label="y = x^2", linestyle='-')
plt.plot(x, z, "r", label="z = 5x + 7", linestyle='-.')
plt.legend(loc="upper left")
plt.xlabel("x")
plt.ylabel("f(x)")
plt.title("An Interesting Graph")
plt.savefig('plot.png')
```



Probability by Simulation

P(E)

The long-term limit of probability of an event E occurring in a random experiment

$$\frac{\#\ of\ trials\ (E)}{\#trials} \rightarrow P(E)$$

A Coin Flip Game

Suppose a weighted coin comes up heads with probability $\frac{1}{3}$.

How many flips do you think it will take for the first head to appear?

Simulating the Coin Flip Game

```
np.random.rand()
```

→

```
Returns a single
random float in
the range [0, 1)
```

Simulating the Coin Flip Game

```
if np.random.rand() < p:
```

What is this expression checking?

Since np.random.rand() returns a random float between [0, 1), the function returns a value < p with probability p.

Simulating the Coin Flip Game

```
if np.random.rand() < p:
```

What is this expression checking?

Since np.random.rand() returns a random float between [0, 1), the function returns a value < p with probability p.

This allows us to simulate the event in question: **the first ‘Heads’ appears whenever rand() returns a value < p.**
And, if rand() >= p, the coin flip turned up ‘Tails’.

Simulating ONE Coin Flip Game

```
def sim_one_game():
    flips = 0
    while True:
        flips += 1
        if np.random.rand() < p:
            return flips
```

Counter that keeps track of number of coin flips

When we “flip a head”, we return the total number of times we’ve flipped the coin.

P(heads)

Number of games we want to simulate

Helper function simulates one game

```
import numpy as np

def coin_flips(p:float = 1/3, ntrials:int = 5000) -> float:
    def sim_one_game() -> int:
        flips = 0
        while True:
            flips += 1
            if np.random.rand() < p:
                return flips

    total_flips = 0
    for i in range(ntrials):
        total_flips += sim_one_game()
    return total_flips / ntrials
```

P(heads)

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def coin_flips(p:float = 1/3, ntrials:int = 5000) -> float:
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        flips = 0
        while True:
            flips += 1
            if np.random.rand() < p:
                return flips

    total_flips = 0
    for i in range(ntrials):
        total_flips += sim_one_game()
    return total_flips / ntrials
```

After each game, adds the total number of flips taken.

Finally, we return the average # of flips it took for the first H to appear

Codealong: Probability via Simulation