

Section 7

Review

1) **Normal (Gaussian, “bell curve”)**: $X \sim \mathcal{N}(\mu, \sigma^2)$ iff X has the following probability density function:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}, \quad x \in \mathbb{R}$$

$\mathbb{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2$. The “standard normal” random variable is typically denoted Z and has mean 0 and variance 1: if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$. The CDF has no closed form, but we denote the CDF of the standard normal as $\Phi(z) = F_Z(z) = \mathbb{P}(Z \leq z)$. Note from symmetry of the probability density function about $z = 0$ that: $\Phi(-z) = 1 - \Phi(z)$.

Here is the **Standard normal table**.

2) **Standardizing**: Let X be any random variable (discrete or continuous, not necessarily normal), with $\mathbb{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2$. If we let $Y = \frac{X-\mu}{\sigma}$, then $\mathbb{E}[Y] = 0$ and $\text{Var}(Y) = 1$.

3) **Closure of the Normal Distribution**: Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then, $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$. That is, linear transformations of normal random variables are still normal.

4) **“Reproductive” Property of Normals**: Let X_1, \dots, X_n be independent normal random variables with $\mathbb{E}[X_i] = \mu_i$ and $\text{Var}(X_i) = \sigma_i^2$. Let $a_1, \dots, a_n \in \mathbb{R}$ and $b \in \mathbb{R}$. Then,

$$X = \sum_{i=1}^n (a_i X_i + b) \sim \mathcal{N}\left(\sum_{i=1}^n (a_i \mu_i + b), \sum_{i=1}^n a_i^2 \sigma_i^2\right)$$

There’s nothing special about the parameters – the important result here is that the resulting random variable is still normally distributed.

5) **Law of Total Probability (Continuous)**: A is an event, and X is a continuous random variable with density function $f_X(x)$.

$$\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A|X=x) f_X(x) dx$$

6) **Central Limit Theorem (CLT)**: Let X_1, \dots, X_n be iid random variables with $\mathbb{E}[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$. Let $X = \sum_{i=1}^n X_i$, which has $\mathbb{E}[X] = n\mu$ and $\text{Var}(X) = n\sigma^2$. Let $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$, which has $\mathbb{E}[\bar{X}] = \mu$ and $\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$. \bar{X} is called the *sample mean*. Then, as $n \rightarrow \infty$, \bar{X} approaches the normal distribution $\mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$. Standardizing, this is equivalent to $Y = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ approaching $\mathcal{N}(0, 1)$. Similarly, as $n \rightarrow \infty$, X approaches $\mathcal{N}(n\mu, n\sigma^2)$ and $Y' = \frac{X-n\mu}{\sigma\sqrt{n}}$ approaches $\mathcal{N}(0, 1)$.

It is no surprise that \bar{X} has mean μ and variance σ^2/n – this can be done with simple calculations. The importance of the CLT is that, for large n , regardless of what distribution X_i comes from, \bar{X} is *approximately normally distributed with mean μ and variance σ^2/n* . Don’t forget the continuity correction, only when X_1, \dots, X_n are discrete random variables.

7) **Multivariate: Discrete to Continuous:** To be discussed next week....

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x, y) = \mathbb{P}(X = x, Y = y)$	$f_{X,Y}(x, y) \neq \mathbb{P}(X = x, Y = y)$
Joint range/support $\Omega_{X,Y}$	$\{(x, y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x, y) > 0\}$	$\{(x, y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x, y) > 0\}$
Joint CDF	$F_{X,Y}(x, y) = \sum_{t \leq x, s \leq y} p_{X,Y}(t, s)$	$F_{X,Y}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(t, s) ds dt$
Normalization	$\sum_{x,y} p_{X,Y}(x, y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x, y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$
Expectation	$\mathbb{E}[g(X, Y)] = \sum_{x,y} g(x, y) p_{X,Y}(x, y)$	$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx dy$
Independence must have	$\forall x, y, p_{X,Y}(x, y) = p_X(x)p_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$	$\forall x, y, f_{X,Y}(x, y) = f_X(x)f_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$

Task 1 – Normal questions at the table (repeat)

- a) Let X be a normal random with parameters $\mu = 10$ and $\sigma^2 = 36$. Compute $\mathbb{P}(4 < X < 16)$.
- b) Let X be a normal random variable with mean 5. If $\mathbb{P}(X > 9) = 0.2$, approximately what is $\text{Var}(X)$?
- c) Let X be a normal random variable with mean 12 and variance 4. Find the value of c such that $\mathbb{P}(X > c) = 0.10$.

Task 2 – Round-off error

Let X be the sum of 100 real numbers, and let Y be the same sum, but with each number rounded to the nearest integer before summing. If the roundoff errors are independent and uniformly distributed between -0.5 and 0.5, what is the approximate probability that $|X - Y| > 3$?

Task 3 – Tweets

A prolific Twitter user tweets approximately 350 tweets per week. Let's assume for simplicity that the tweets are independent, and each consists of a uniformly random number of characters between 10 and 140. (Note that this is a discrete uniform distribution.) Thus, the central limit theorem (CLT) implies that the number of characters tweeted by this user is approximately normal with an appropriate mean and variance. Assuming this normal approximation is correct, estimate the probability that this user tweets between 26,000 and 27,000 characters in a particular week. (This is a case where continuity correction will make virtually no difference in the answer, but you should still use it to get into the practice!).

Task 4 – Confidence interval

Suppose that X_1, \dots, X_n are i.i.d. samples from a normal distribution with unknown mean μ and variance 36. How big does n need to be so that μ is in

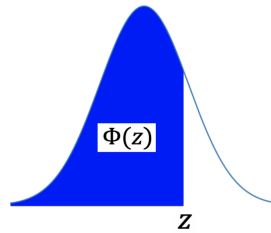
$$[\bar{X} - 0.11, \bar{X} + 0.11]$$

with probability at least 0.97?

Recall that

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i.$$

You may use the fact that $\Phi^{-1}(0.985) = 2.17$.



Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0, 1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999