CSE 312

Foundations of Computing II

Lecture 14: Quick wrapup of discrete RVs
Continuous RV

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Agenda

- Wrap-up of Discrete RVs
- Continuous Random Variables
- Probability Density Function
- Cumulative Distribution Function
- Expectation and Variance of continuous RVs

Poisson Random Variables

Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

General principle:

- Events happen at an average rate of λ per time unit
- Number of events happening at a time unit X is distributed according to Poi(λ)

- Poisson approximates Binomial when n is large, p is small, and np is moderate
- Sum of independent Poisson is still a Poisson

Zoo of Random Variables 🦍🐘🦁🐅🦓🐪🦒

$X \sim \text{Poisson}(\lambda)$

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

$$E[X] = \lambda$$

$$\text{Var}(X) = \lambda$$

$X \sim \text{Ber}(p)$

$$P(X = 1) = p, P(X = 0) = 1 - p$$

$$E[X] = p$$

$$\text{Var}(X) = p(1 - p)$$

$X \sim \text{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$E[X] = np$$

$$\text{Var}(X) = np(1 - p)$$

$X \sim \text{Geo}(p)$

$$P(X = k) = (1 - p)^{k-1} p$$

$$E[X] = \frac{1}{p}$$

$$\text{Var}(X) = \frac{1 - p}{p^2}$$

$X \sim \text{NegBin}(r, p)$

$$P(X = k) = \binom{k - 1}{r - 1} p^r (1 - p)^{k-r}$$

$$E[X] = \frac{r}{p}$$

$$\text{Var}(X) = \frac{r(1 - p)}{p^2}$$

$X \sim \text{HypGeo}(N, K, n)$

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

$$E[X] = n\frac{K}{N}$$

$$\text{Var}(X) = n\frac{K(N - K)(N - n)}{N^2(N - 1)}$$

Negative Binomial Random Variables

A discrete random variable X that models the number of independent trials $Y_i \sim \mathrm{Ber}(p)$ before seeing the r^{th} success.
Equivalently, $X = \sum_{i=1}^{r} Z_i$ where $\mathrm{Z}_i \sim \mathrm{Geo}(p)$.
X is called a Negative Binomial random variable with parameters r, p.

Notation: $X \sim \mathrm{NegBin}(r, p)$

PMF: $P(X = k) =$

Expectation: $\mathbb{E}[X] =$

Negative Binomial Random Variables

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X is called a Negative Binomial random variable with parameters r, p.

Notation: $X \sim \mathrm{NegBin}(r, p)$

PMF: $P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$

Expectation: $\mathbb{E}[X] = \frac{r}{p}$

Variance: $\mathrm{Var}(X) = \frac{r(1-p)}{p^2}$

Hypergeometric Random Variables

A discrete random variable X that models the number of successes in n draws (without replacement) from N items that contain K successes in total. X is called a Hypergeometric RV with parameters N, K, n.

Notation: $X \sim \mathrm{HypGeo}(N, K, n)$

PMF: $P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$

Expectation: $\mathbb{E}[X] = n\frac{K}{N}$

Variance: $\mathrm{Var}(X) = n\frac{K(N-K)(N-n)}{N^2(N-1)}$

Hope you enjoyed the zoo! 🦍🐘🦁🐅🦓🐪🦒

$X \sim \text{Unif}(a, b)$

$$P(X = k) = \frac{1}{b - a + 1}$$

$$\mathbb{E}[X] = \frac{a + b}{2}$$

$$\text{Var}(X) = \frac{(b - a)(b - a + 2)}{12}$$

$X \sim \text{Ber}(p)$

$$P(X = 1) = p, P(X = 0) = 1 - p$$

$$\mathbb{E}[X] = p$$

$$\text{Var}(X) = p(1 - p)$$

$X \sim \text{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$\mathbb{E}[X] = np$$

$$\text{Var}(X) = np(1 - p)$$

$X \sim \text{Geo}(p)$

$$P(X = k) = (1 - p)^{k-1} p$$

$$\mathbb{E}[X] = \frac{1}{p}$$

$$\text{Var}(X) = \frac{1 - p}{p^2}$$

$X \sim \text{NegBin}(r, p)$

$$P(X = k) = \binom{k - 1}{r - \ldots}$$

$$\mathbb{E}[X] = \frac{r}{p}$$

$$\text{Var}(X) = \frac{r(1 - \ldots}{p^2}$$

$X \sim \text{HypGeo}(N, K, n)$

$$\frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

$$\frac{\ldots(N - K)(N - n)}{N^2(N - 1)}$$

$X \sim \text{Poisson}(\lambda)$

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

$$E[X] = \lambda$$

$$\text{Var}(X) = \lambda$$

Agenda

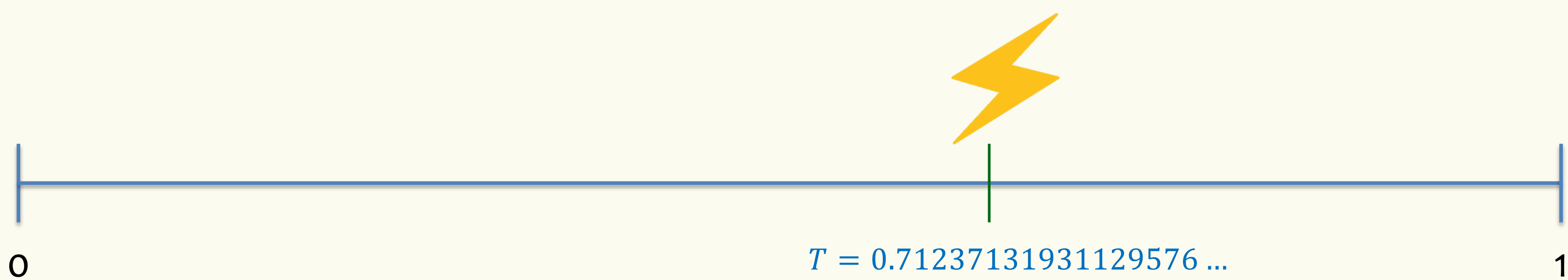
- Wrap-up of Poisson RVs
- Continuous Random Variables
- Probability Density Function
- Cumulative Distribution Function
- Expectation and Variance of continuous r.v.

Often we want to model experiments where the outcome is <u>not</u> discrete.

Example – Lightning Strike

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every time within $[0,1]$ is equally likely
  - – Time measured with infinitesimal precision.

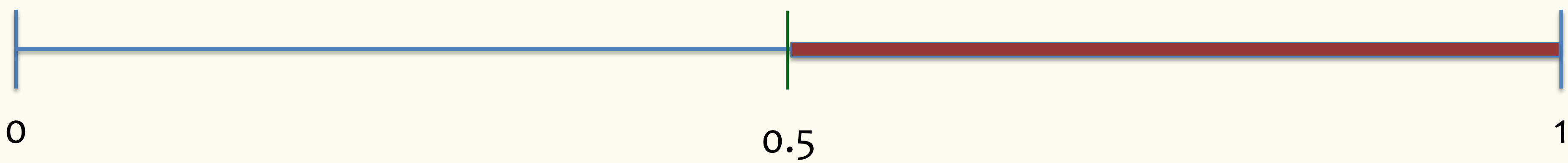


The outcome space is not discrete

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely

$$P(T \geq 0.5) =$$



Lightning strikes a pole within a one-minute time frame

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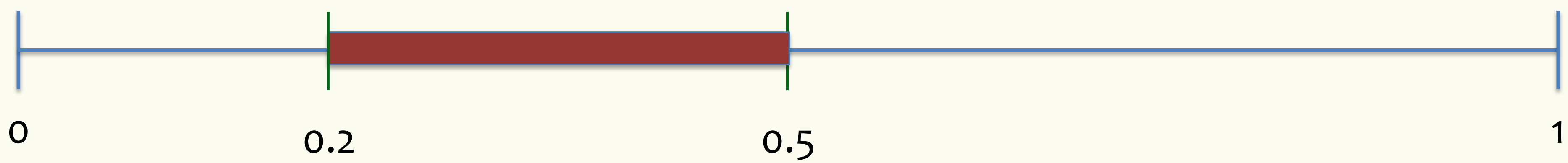
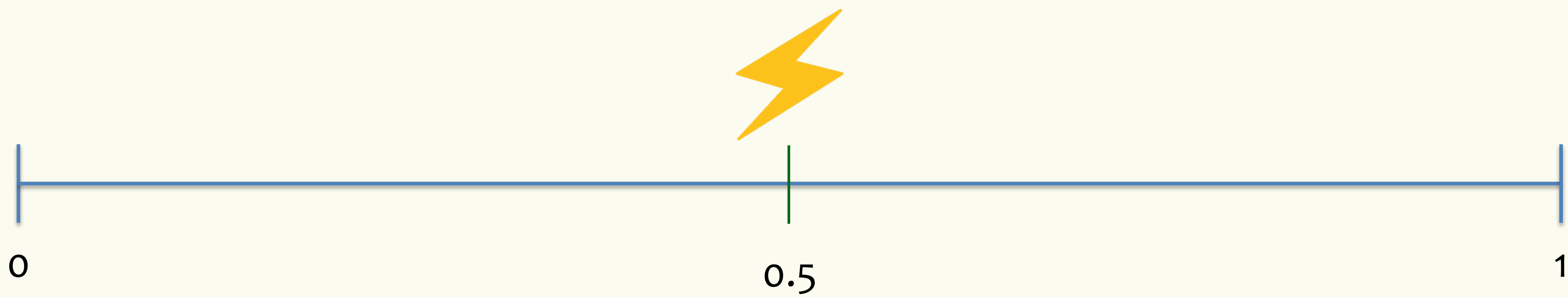


$P(0.2 \le T \le 0.5) =$

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely



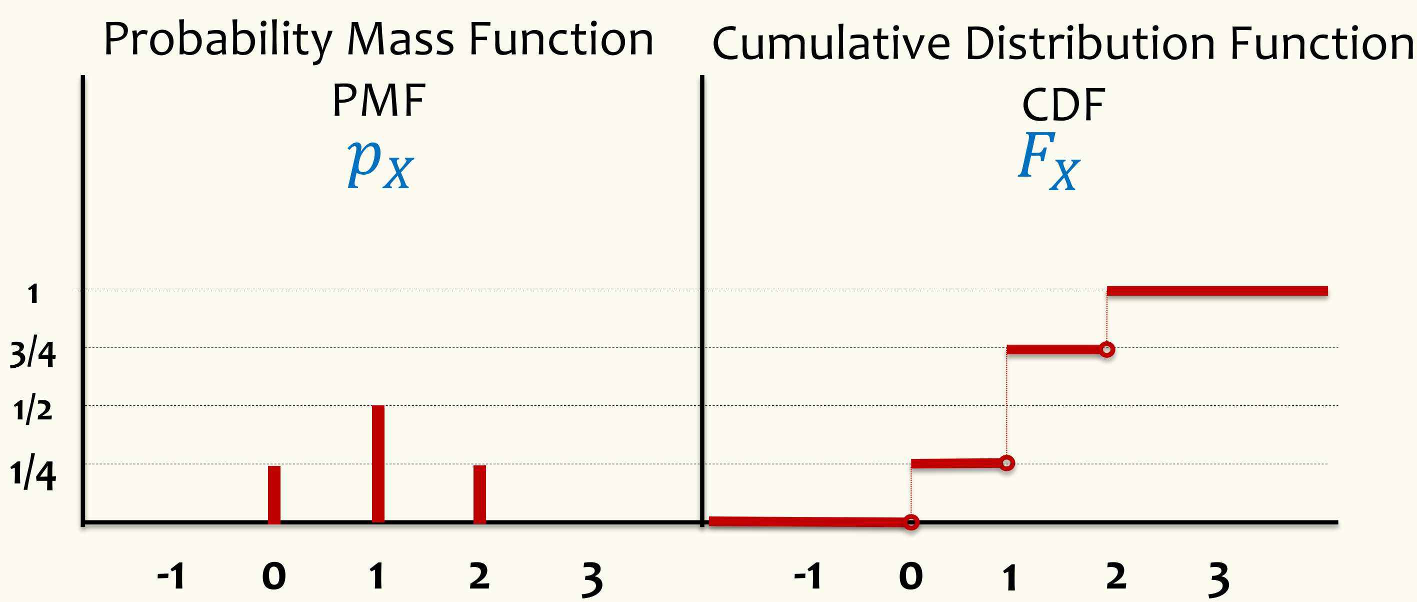
$P(T = 0.5) =$

Bottom line

- This gives rise to a different type of random variable
- $P(T = x) = 0$ for all $x \in [0,1]$
- Yet, somehow we want
  - $P(T \in [0,1]) = 1$
  - $P(T \in [a, b]) = b - a$
  - …
- How do we model the behavior of T?

First try: A discrete approximation

Recall: Cumulative Distribution Function (CDF)



Poll: Given the CDF, how do you compute the pmf?

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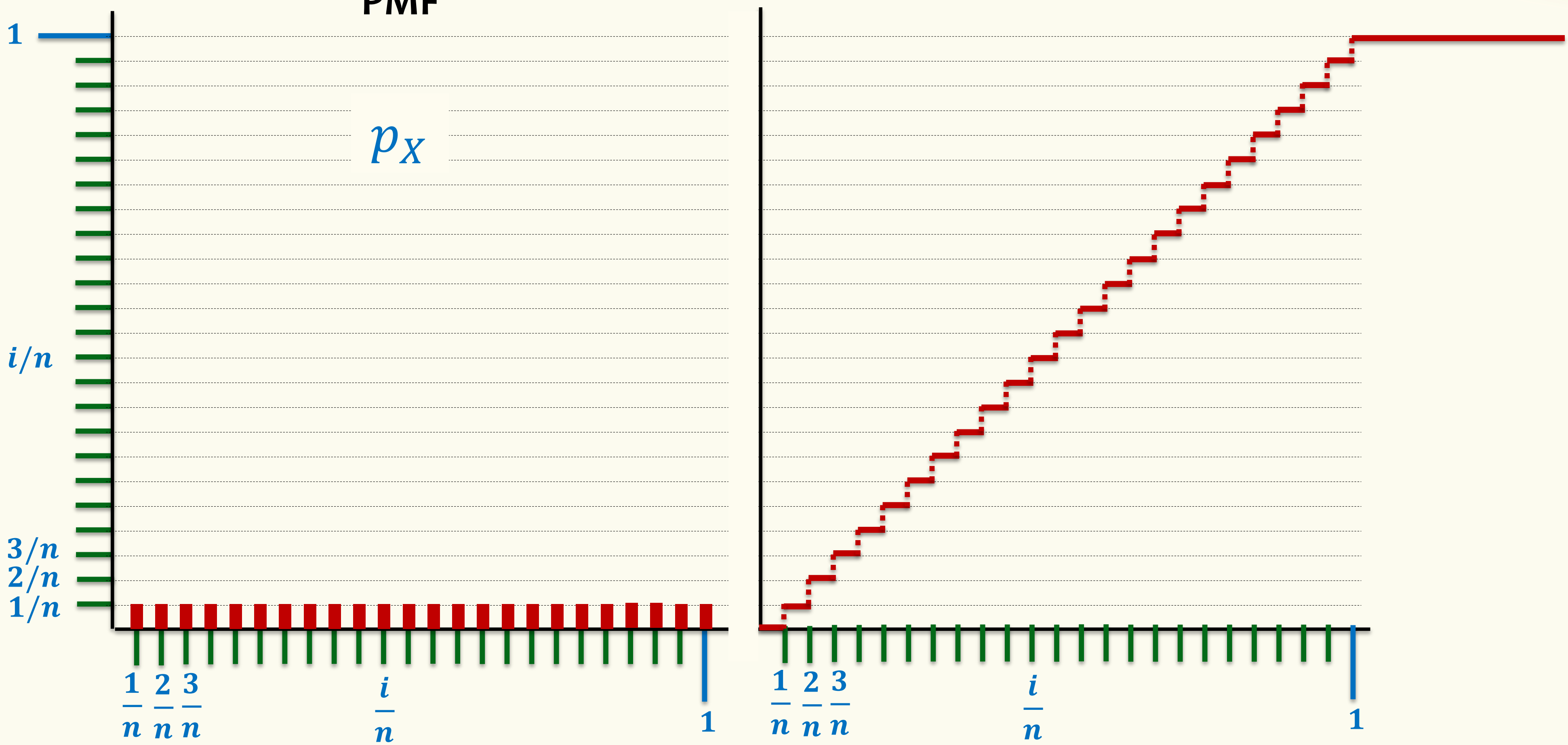
$\Pr(X = k) =$

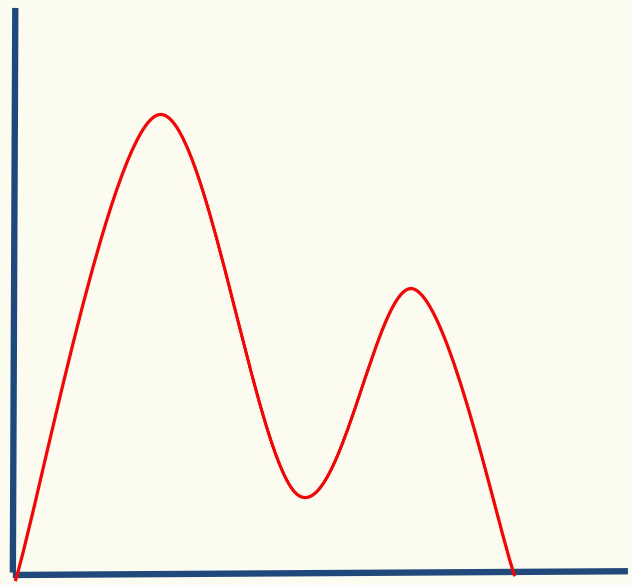
a. $F_X(k-1)$
b. $F_X(1) + F_X(2) + \cdots + F_X(k-1)$
c. $F_X(k) - F_X(k-1)$
d. I don't know.

A Discrete Approximation

Probability Mass Function
PMF

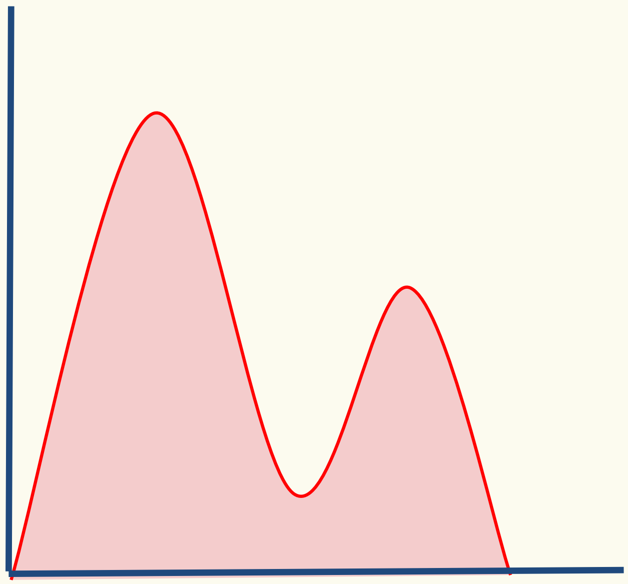


Definition. A **continuous random variable** X is defined by a **probability density function** (PDF) $f_X : \mathbb{R} \to \mathbb{R}$, such that



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

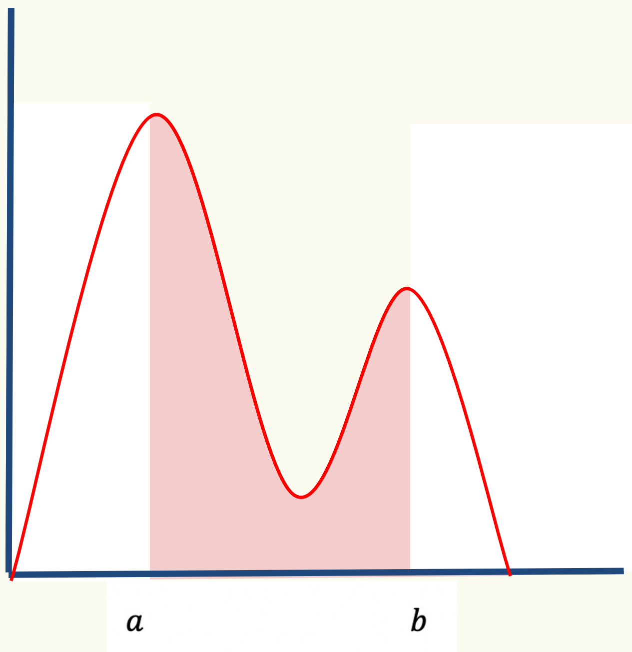
Probability Density Function - Intuition



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

Probability Density Function - Intuition



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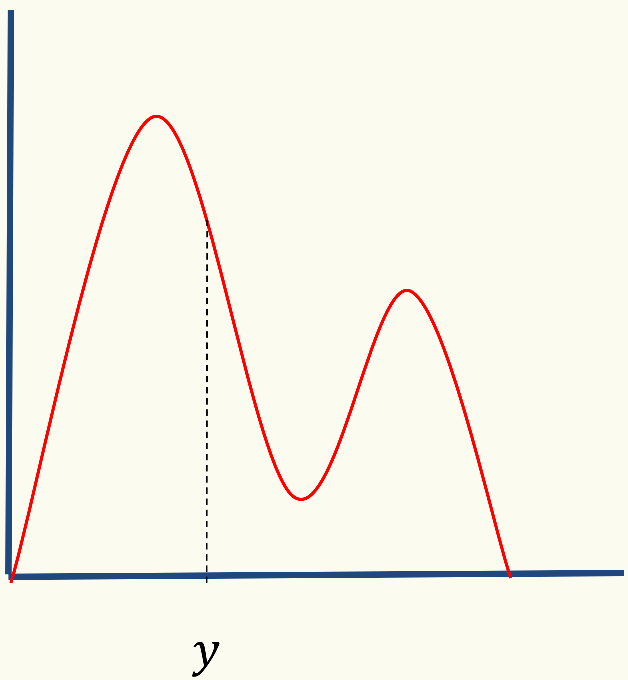
$$P(a \leq X \leq b) = \int_a^b f_X(x)\, \mathrm{d}x$$

Probability Density Function - Intuition



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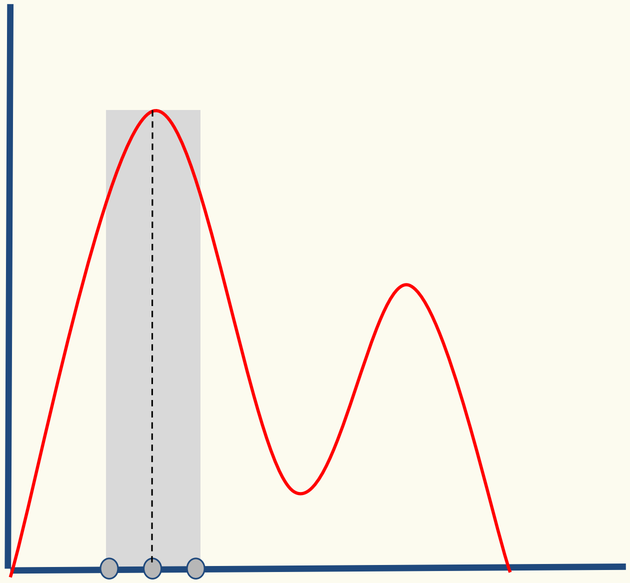
$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

Density ≠ Probability

$f_X(y) \neq 0 \qquad P(X = y) = 0$

Probability Density Function - Intuition



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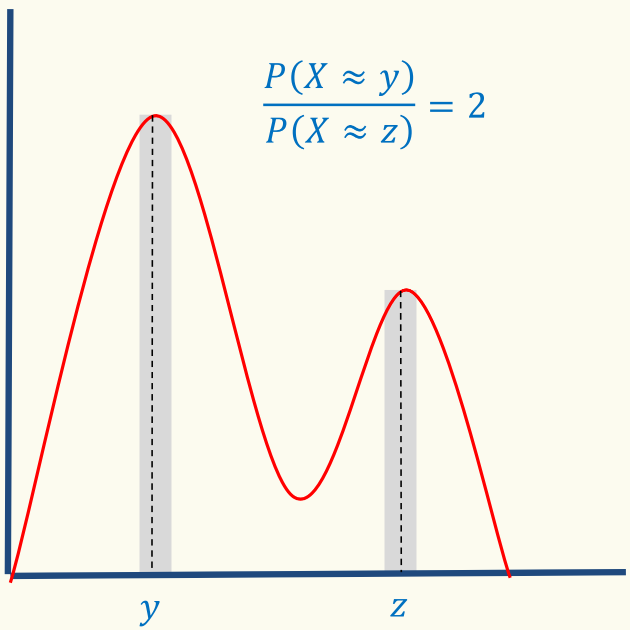
$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\, \mathrm{d}x = 0$$

$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x)\, \mathrm{d}x \approx \epsilon f_X(y)$$

What $f_X(x)$ measures: The local ***rate*** at which probability accumulates

Probability Density Function - Intuition



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$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$

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$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$

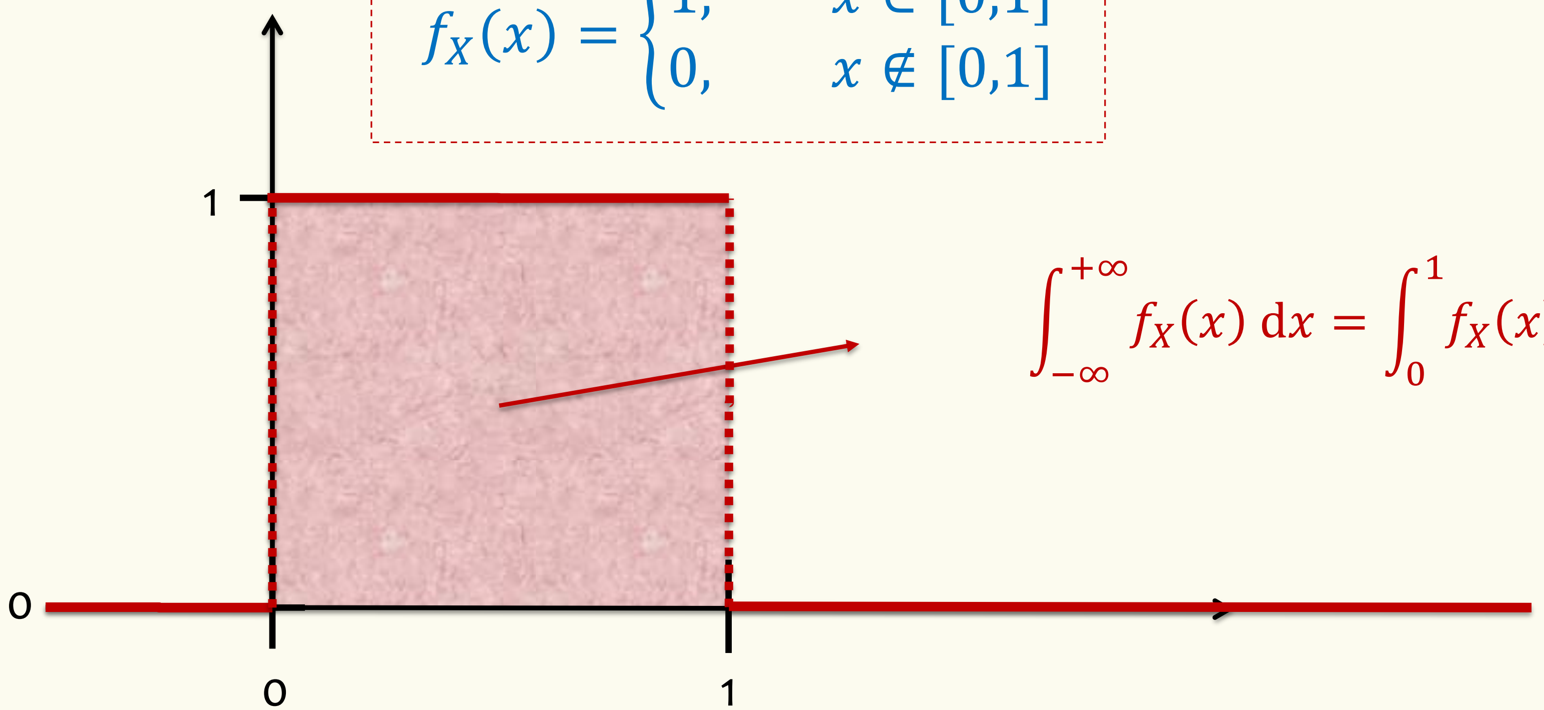
PDF of Uniform RV

$X \sim \mathrm{Unif}(0,1)$

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = \int_0^1 f_X(x)\,\mathrm{d}x = 1 \cdot 1 = 1$$

Probability of Event

$X \sim \mathrm{Unif}(0,1)$

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

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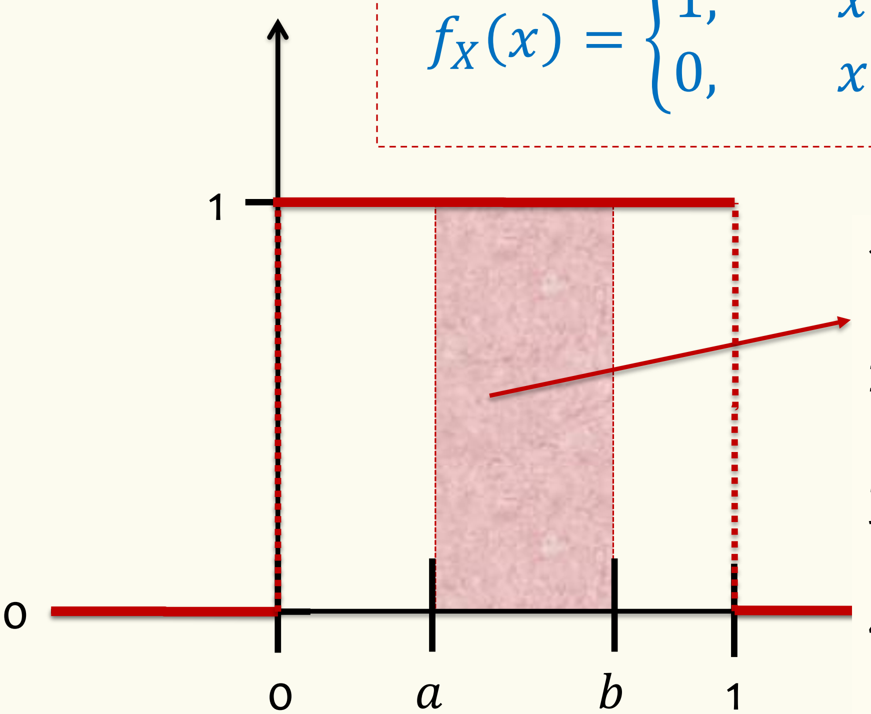
$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$



1. If $0 \leq a$ and $a \leq b \leq 1$
   $P(a \leq X \leq b) = b - a$
2. If $a < 0$ and $0 \leq b \leq 1$
   $P(a \leq X \leq b) = b$
3. If $a \geq 0$ and $b > 1$
   $P(a \leq X \leq b) = b - a$
4. If $a < 0$ and $b > 1$
   $P(a \leq X \leq b) = 1$

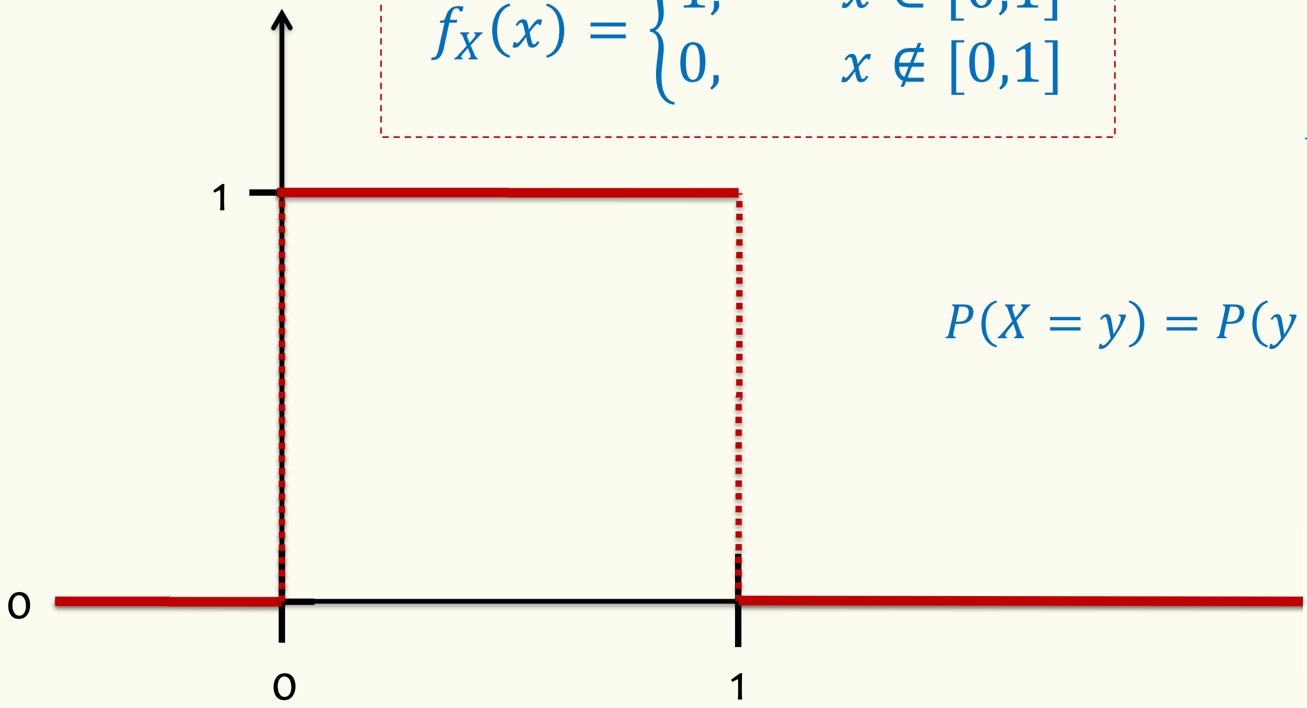
Poll: Slido.com/4694375

A. All of them are correct
B. Only 1, 2, 4 are right
C. Only 1 is right
D. Only 1 and 2 are right

Probability of Event

$X \sim \text{Unif}(0,1)$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



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Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

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$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

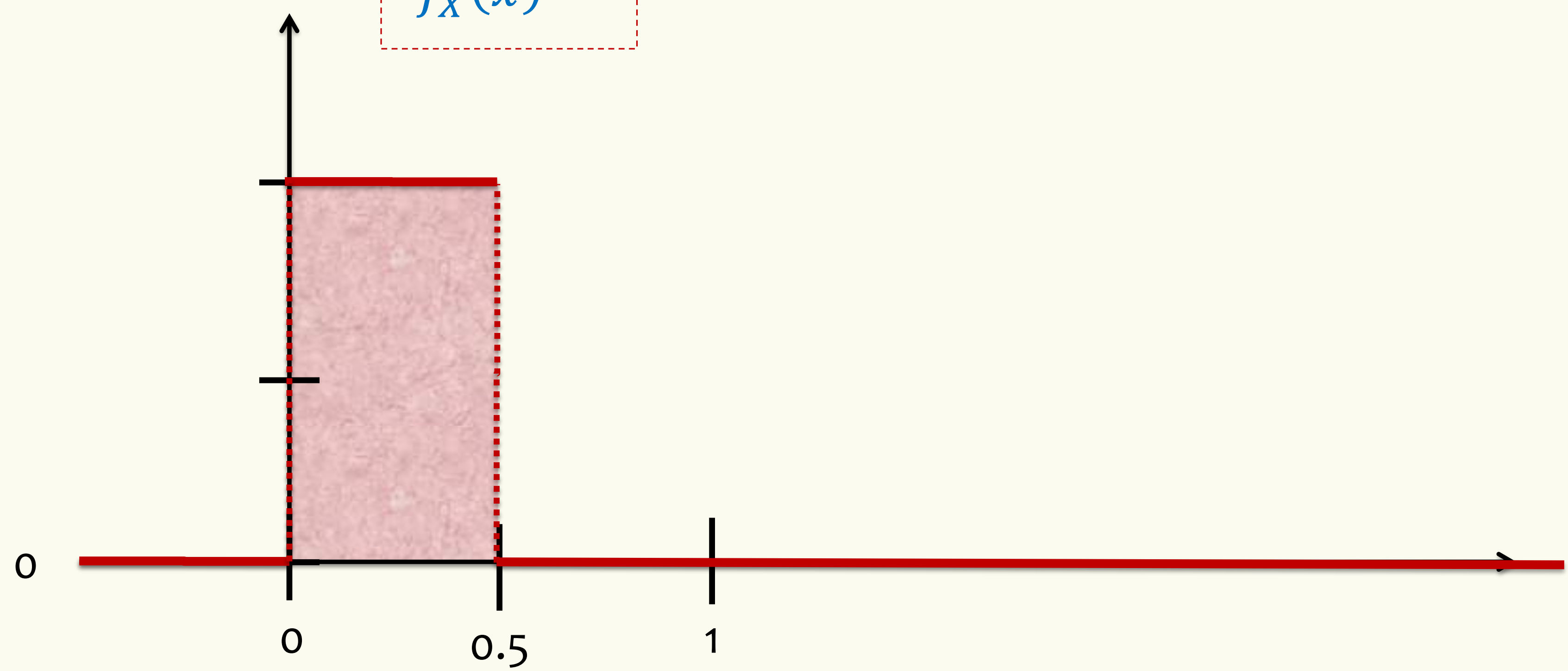
$$P(X \approx y) \approx \epsilon f_X(y) = \epsilon$$

$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$

PDF of Uniform RV

$X \sim \mathrm{Unif}(0,0.5)$

$f_X(x) =$



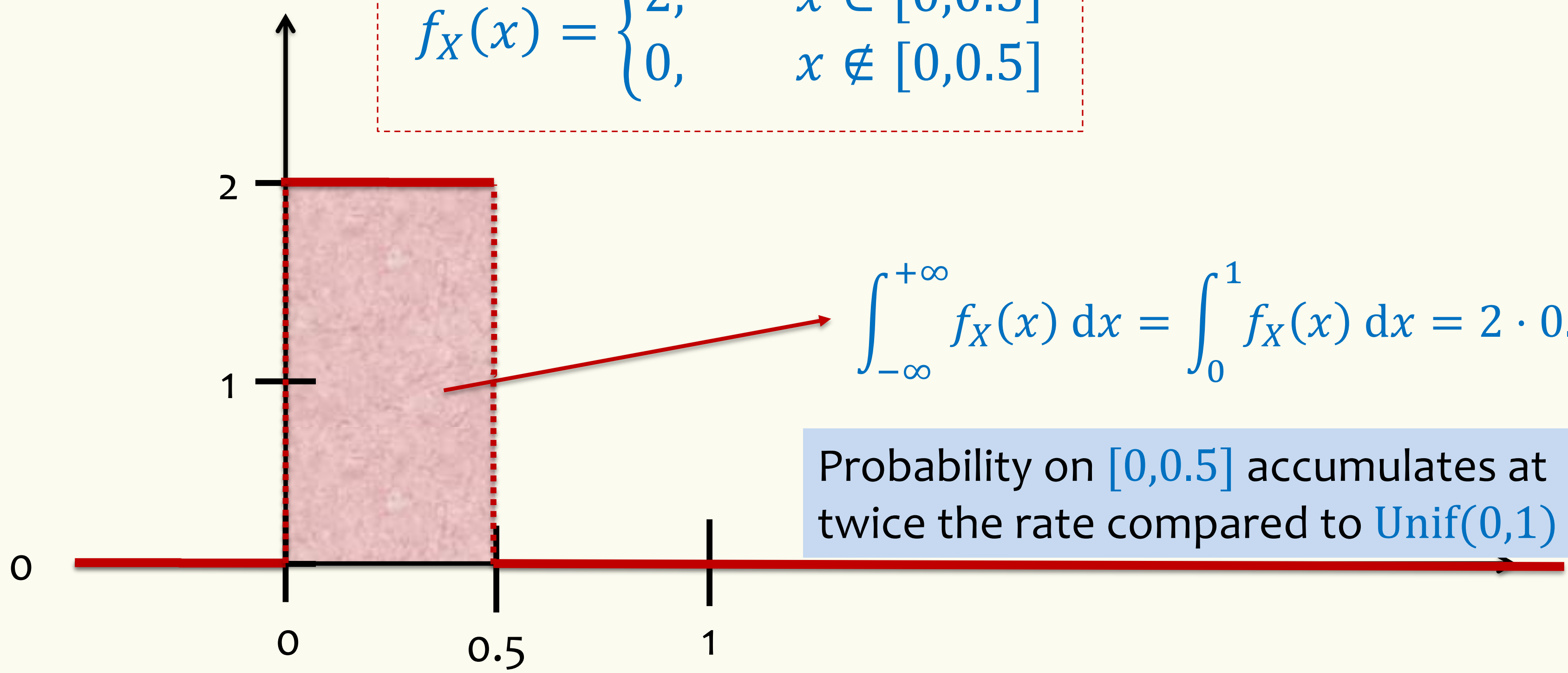
PDF of Uniform RV

Density $\neq$ Probability

$f_X(x) \gg 1$ is possible!

$X \sim \text{Unif}(0,0.5)$

$$f_X(x) = \begin{cases} 2, & x \in [0,0.5] \\ 0, & x \notin [0,0.5] \end{cases}$$



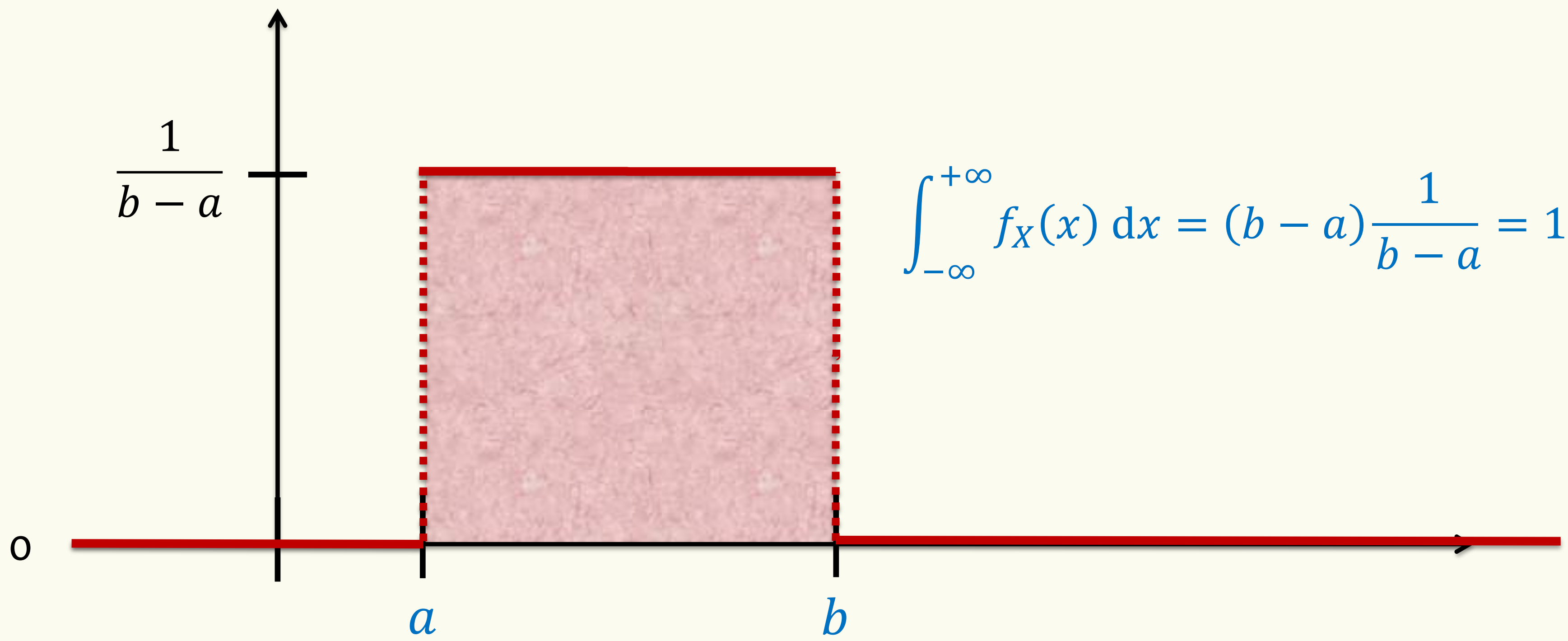
$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = \int_0^1 f_X(x)\,\mathrm{d}x = 2 \cdot 0.5 = 1$$

Probability on $[0,0.5]$ accumulates at twice the rate compared to $\text{Unif}(0,1)$

Uniform Distribution

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$X \sim \mathrm{Unif}(a, b)$

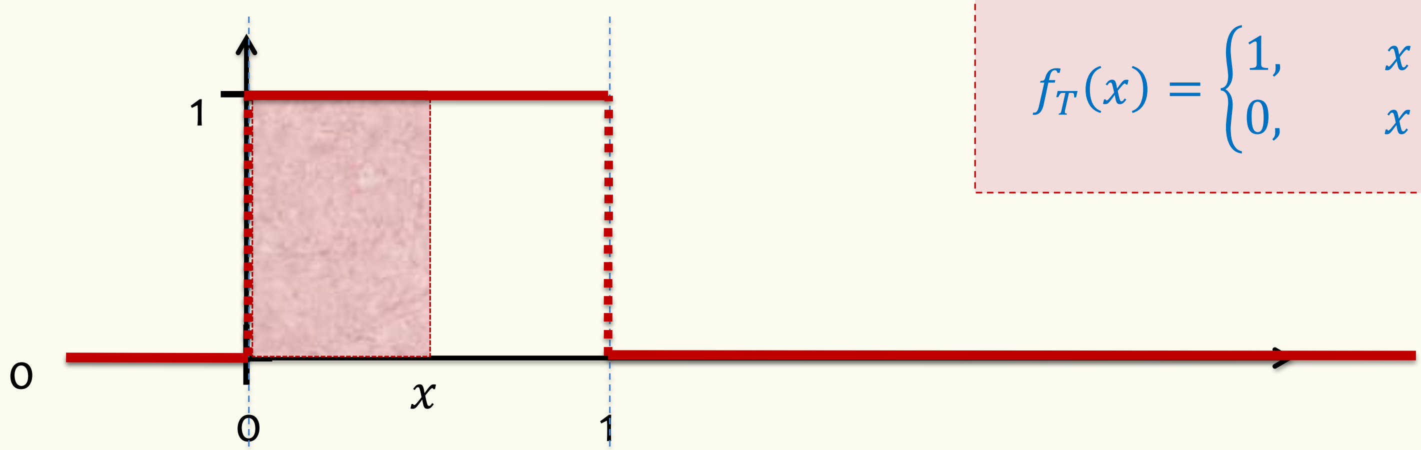


$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = (b-a)\frac{1}{b-a} = 1$$

Example. $T \sim \text{Unif}(0,1)$

Probability Density Function

$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



Cumulative Distribution Function

$$F_T(x) = P(T \leq x) = \begin{cases} 0 & x \leq 0 \\ ? & 0 \leq x \leq 1 \\ 1 & 1 \leq x \end{cases}$$

Cumulative Distribution Function

Definition. The **cumulative distribution function (cdf)** of X is

$$F_X(a) = P(X \leq a) = \int_{-\infty}^{a} f_X(x)\,\mathrm{d}x$$

By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx} F_X(x)$

Cumulative Distribution Function

Definition. The **cumulative distribution function (cdf)** of X is

$$F_X(a) = P(X \leq a) = \int_{-\infty}^{a} f_X(x)\,\mathrm{d}x$$

By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx} F_X(x)$

Therefore: $P(X \in [a, b]) = F_X(b) - F_X(a)$

F_X is monotone increasing, since $f_X(x) \geq 0$. That is $F_X(c) \leq F_X(d)$ for $c \leq d$

$\lim_{a \to -\infty} F_X(a) = P(X \leq -\infty) = 0$ $\quad$ $\lim_{a \to +\infty} F_X(a) = P(X \leq +\infty) = 1$

From Discrete to Continuous

	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_{x} p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
Expectation	$\mathbb{E}[g(X)] = \sum_{x} g(x)\, p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$

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- Wrap-up of Poisson RVs
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Expectation of a Continuous RV

Definition. The **expected value** of a continuous RV X is defined as

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

Fact. $\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$

Proof follows same ideas as discrete case

Expectation of a Continuous RV

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Fact. $\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$

Proofs follow same ideas as discrete case

Definition. The **variance** of a continuous RV X is defined as

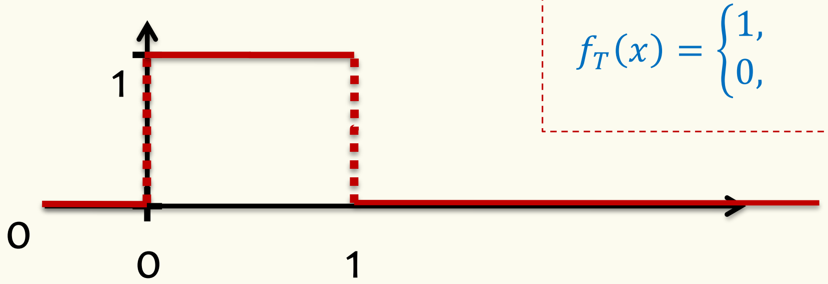
$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot (x - \mathbb{E}[X])^2 \, \mathrm{d}x = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Expectation of a Continuous RV

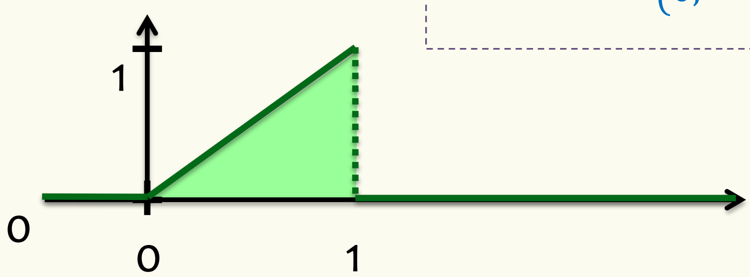
Definition.

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

Example. $T \sim \mathrm{Unif}(0,1)$



$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



$$f_T(x) \cdot x = \begin{cases} x, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

$$\mathbb{E}[T] = \underbrace{\frac{1}{2} 1^2}_{\text{Area of triangle}} = \frac{1}{2}$$

Uniform Density – Expectation

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$X \sim \text{Unif}(a, b)$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

$$= \frac{1}{b-a} \int_a^b x \, \mathrm{d}x \quad = \frac{1}{b-a} \left(\frac{x^2}{2} \right) \Bigg|_a^b = \frac{1}{b-a} \left(\frac{b^2 - a^2}{2} \right)$$

$$= \frac{(b-a)(a+b)}{2(b-a)} \quad = \frac{a+b}{2}$$

Uniform Density – Expectation

$X \sim \mathrm{Unif}(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$\mathbb{E}[X] =$

Uniform Density – Variance

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$X \sim \mathrm{Unif}(a,b)$

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 \, \mathrm{d}x$$

$$= \frac{1}{b-a} \int_a^b x^2 \, \mathrm{d}x = \frac{1}{b-a} \left(\frac{x^3}{3} \right) \Bigg|_a^b = \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

Uniform Density – Variance

$X \sim \mathrm{Unif}(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$\mathbb{E}[X^2] =$

Uniform Density – Variance

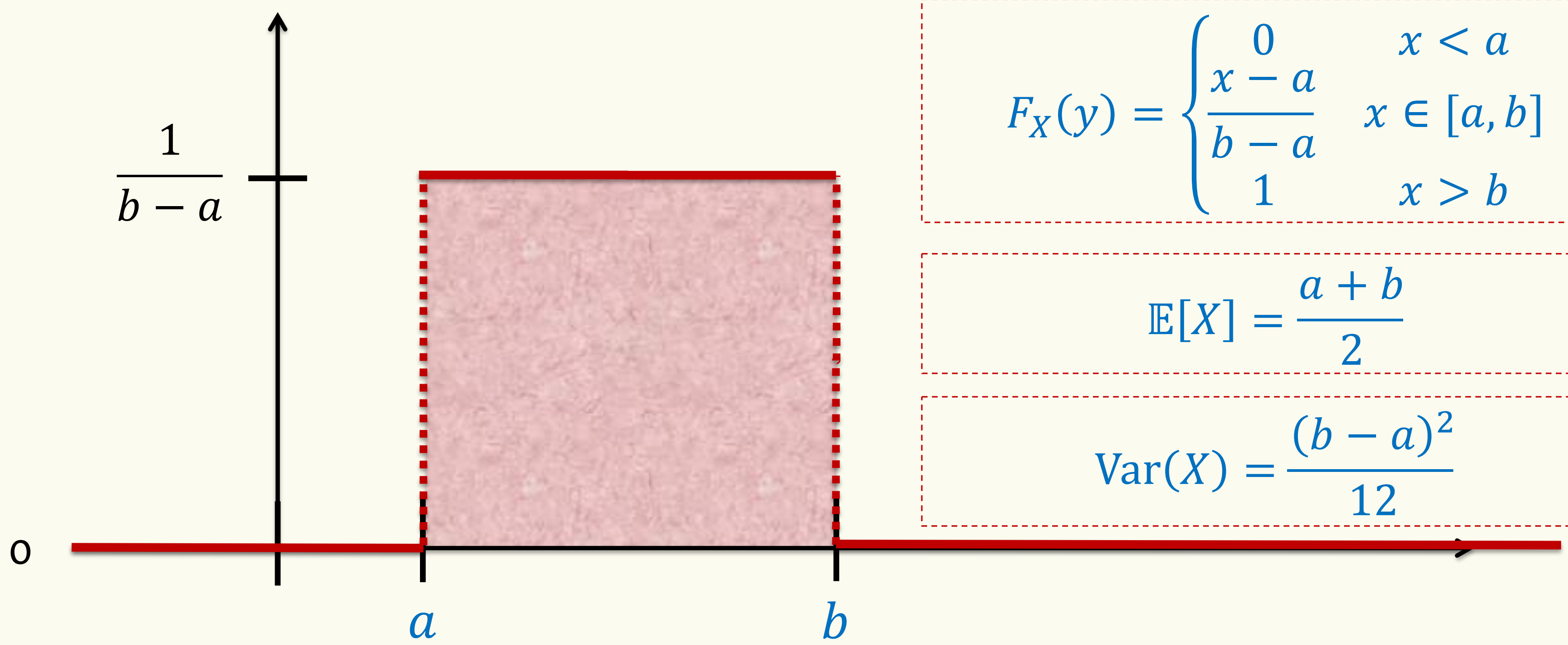
$$\mathbb{E}[X^2] = \frac{b^2 + ab + a^2}{3} \qquad \mathbb{E}[X] = \frac{a+b}{2}$$

$X \sim \text{Unif}(a, b)$

$$\begin{aligned}\text{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ &= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4} \\ &= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12} \\ &= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}\end{aligned}$$

Uniform Distribution Summary

$X \sim \mathrm{Unif}(a, b)$



$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$F_X(y) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & x \in [a, b] \\ 1 & x > b \end{cases}$$

$$\mathbb{E}[X] = \frac{a+b}{2}$$

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$$