CSE 312

# Foundations of Computing II

## 18: Joint Distributions (+ recap polling)

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## Agenda

- Joint Distributions
  - Cartesian Products
  - Joint PMFs and Joint Range
  - Marginal Distribution
  - Analogues for continuous distributions
  - LOTUS for joint distns
- Recap of polling example.

## Why joint distributions?

- Given all of its user's ratings for different movies, and any preferences you have expressed, Netflix wants to recommend a new movie for you.
- Given a large amount of medical data correlating symptoms and personal history with diseases, predict what is ailing a person with a particular medical history and set of symptoms.
- Given current traffic, pedestrian locations, weather, lights, etc. decide whether a self-driving car should slow down or come to a stop

## Review Cartesian Product

**Definition.** Let $A$ and $B$ be sets. The **Cartesian product** of $A$ and $B$ is denoted

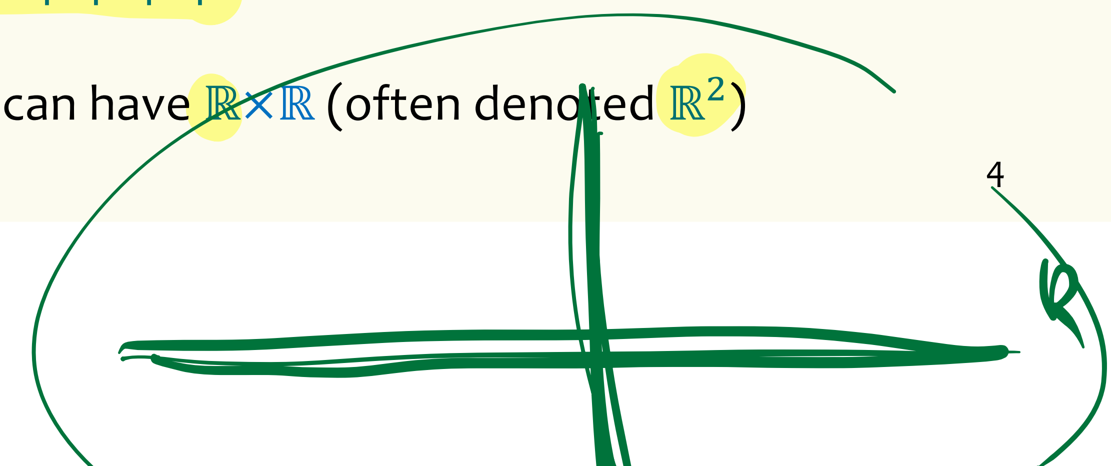
$$A \times B = \{(a, b) : a \in A, b \in B\}$$

**Example.**

$$\{1,2,3\} \times \{4,5\} = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$$

If $A$ and $B$ are finite sets, then $|A \times B| = |A| \cdot |B|$.

The sets don't need to be finite! You can have $\mathbb{R} \times \mathbb{R}$ (often denoted $\mathbb{R}^2$)



## Joint PMFs and Joint Range

$p_X(a) = P(X=a)$

**Definition.** Let $X$ and $Y$ be discrete random variables. The **Joint PMF** of $X$ and $Y$ is

$$p_{X,Y}(a,b) = P(X = a, Y = b)$$

and

**Definition.** The **joint range** of $p_{X,Y}$ is

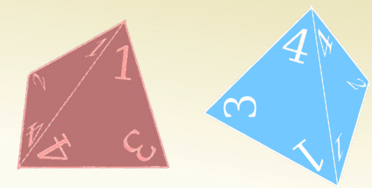
$$\Omega_{X,Y} = \{(c,d) : p_{X,Y}(c,d) > 0\} \subseteq \Omega_X \times \Omega_Y$$

Note that

$$\boxed{\sum_{(s,t)\in\Omega_{X,Y}} p_{X,Y}(s,t) = 1}$$

# Example – Weird Dice

$$p_X(x) = \begin{cases} \frac{1}{4} & x \in \{1,2,3,4\} \\ 0 & o.w. \end{cases}$$



Suppose I roll two fair 4-sided die independently. Let $X$ be the value of the first die, and $Y$ be the value of the second die.

$\Omega_X = \{1,2,3,4\}$ and $\Omega_Y = \{1,2,3,4\}$

In this problem, the joint PMF is if

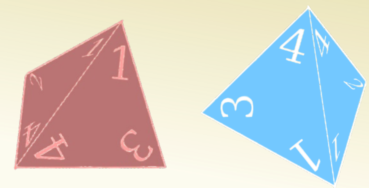
$$p_{X,Y}(x,y) = \begin{cases} 1/16 & \text{if } x, y \in \Omega_{X,Y} \\ 0 & \text{otherwise} \end{cases} = p_X(x)p_Y(y)$$

| X\Y | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1/16 | 1/16 | 1/16 | 1/16 |
| 2 | 1/16 | 1/16 | 1/16 | 1/16 |
| 3 | 1/16 | 1/16 | 1/16 | 1/16 |
| 4 | 1/16 | 1/16 | 1/16 | 1/16 |

and the joint range is (since all combinations have non-zero probability)

$\Omega_{X,Y} = \Omega_X \times \Omega_Y$

# Example – Weirder Dice




Suppose I roll two fair 4-sided die independently. Let $X$ be the value of the first die, and $Y$ be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

$\Omega_U = \{1,2,3,4\}$ and $\Omega_W = \{1,2,3,4\}$

$\Omega_{U,W} = \{(u, w) \in \Omega_U \times \Omega_W : u \leq w\} \neq \boxed{\Omega_U \times \Omega_W}$

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What is $p_{U,W}(1,3) = P(U = 1, W = 3)$?

min max

a. $1/16$
b. $2/16$
c. $1/2$
d. Not sure

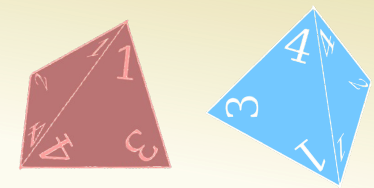
max

min

| U\W | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | | | $\boxed{\frac{2}{16}}$ | |
| 2 | 0 | | | |
| 3 | | | | |
| 4 | | | | |

| X | Y |
|---|---|
| 1 | 3 |
| 3 | 1 |

# Example – Weirder Dice



Suppose I roll two fair 4-sided die independently. Let $X$ be the value of the first die, and $Y$ be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

$\Omega_U = \{1,2,3,4\}$ and $\Omega_W = \{1,2,3,4\}$

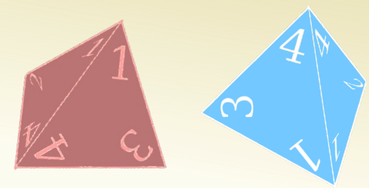
$\Omega_{U,W} = \{(u, w) \in \Omega_U \times \Omega_W : u \leq w\} \neq \Omega_U \times \Omega_W$

The joint PMF $p_{U,W}(u, w) = P(U = u, W = w)$ is

$$p_{U,W}(u,w) = \begin{cases} 2/16 & \text{if } (u,w) \in \Omega_U \times \Omega_W \text{ where } w > u \\ 1/16 & \text{if } (u,w) \in \Omega_U \times \Omega_W \text{ where } w = u \\ 0 & \text{otherwise} \end{cases}$$

| U\W | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1/16 | 2/16 | 2/16 | 2/16 |
| 2 | 0 | 1/16 | 2/16 | 2/16 |
| 3 | 0 | 0 | 1/16 | 2/16 |
| 4 | 0 | 0 | 0 | 1/16 |

# Example – Weirder Dice



Suppose I roll two fair 4-sided die independently. Let $X$ be the value of the first die, and $Y$ be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

Suppose we didn't know how to compute $P(U = u)$ directly. Can we figure it out if we know $p_{U,W}(u, w)$?

$p_U(2) = P(U=2)$

$\frac{5}{16}$ $\frac{7}{16}$

| U\W | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1/16 | 2/16 | 2/16 | 2/16 |
| 2 | 0 | 1/16 | 2/16 | 2/16 |
| 3 | 0 | 0 | 1/16 | 2/16 |
| 4 | 0 | 0 | 0 | 1/16 |

$\frac{7}{16}$

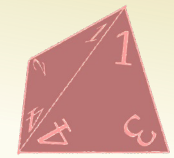
$\frac{5}{16}$

$U=2, V=2$

$U=2, V=3$

$p_{U,V}(u,v)$

# Example – Weirder Dice




Suppose I roll two fair 4-sided die independently. Let $X$ be the value of the first die, and $Y$ be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

Suppose we didn't know how to compute $P(U = u)$ directly. Can we figure it out if we know $p_{U,W}(u, w)$?

Just apply LTP over the possible values of $W$:

$$p_U(1) = 7/16$$

$$p_U(2) = 5/16$$

$$p_U(3) = 3/16$$

$$p_U(4) = 1/16$$

| U\W | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1/16 | 2/16 | 2/16 | 2/16 |
| 2 | 0 | 1/16 | 2/16 | 2/16 |
| 3 | 0 | 0 | 1/16 | 2/16 |
| 4 | 0 | 0 | 0 | 1/16 |

## Marginal PMF

**Definition.** Let $X$ and $Y$ be discrete random variables and $p_{X,Y}(a,b)$ their joint PMF. The **marginal PMF** of $X$

$$p_X(a) = \sum_{b \in \Omega_Y} p_{X,Y}(a,b)$$

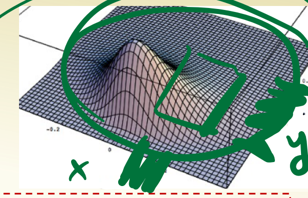
$P(X=a)$ $\quad$ $P(X=a, Y=b)$

Similarly, $p_Y(b) = \sum_{a \in \Omega_X} p_{X,Y}(a,b)$

$$F_{X,Y}(a,b) = P(X \le a, Y \le b)$$

$f_X(x)\,dx$

## Continuous distributions on $\mathbb{R}\times\mathbb{R}$



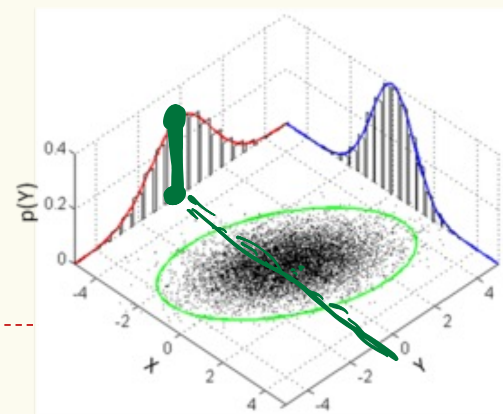
**Definition.** The **joint probability density function (PDF)** of continuous random variables $X$ and $Y$ is a function $f_{X,Y}$ defined on $\mathbb{R}\times\mathbb{R}$ such that

- $f_{X,Y}(x,y) \geq 0$ for all $x, y \in \mathbb{R}$
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\mathrm{d}x\,\mathrm{d}y = 1$

for $A \subseteq \mathbb{R}\times\mathbb{R}$ the probability that $(X,Y) \in A$ is $\iint_A f_{X,Y}(x,y)\,\mathrm{d}x\mathrm{d}y$

The **(marginal) PDFs** $f_X$ and $f_Y$ are given by

- $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y$
- $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x$



# Independence and joint distributions

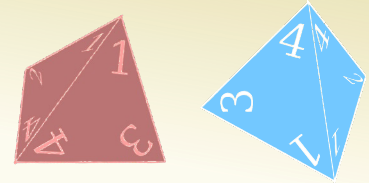
**Definition.** Discrete random variables $X$ and $Y$ are **independent** iff

- $p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y)$ for all $x \in \Omega_X, y \in \Omega_Y$

**Definition.** Continuous random variables $X$ and $Y$ are **independent** iff

- $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$ for all $x, y \in \mathbb{R}$

# Example – Weirder Dice




Suppose I roll two fair 4-sided die independently. Let $X$ be the value of the first die, and $Y$ be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

$\Omega_U = \{1,2,3,4\}$ and $\Omega_W = \{1,2,3,4\}$

$$\Omega_{U,W} = \{(u, w) \in \Omega_U \times \Omega_W : u \leq w\} \neq \Omega_U \times \Omega_W$$

$P(U=u) > 0$

$P(V=v) > 0$

$\boxed{P(U=u, V=v) = 0}$

max

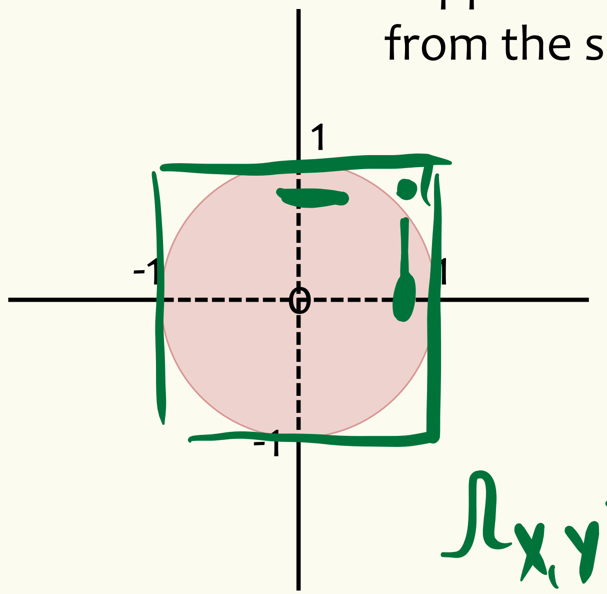
min

| U\W | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| **1** | 1/16 | 2/16 | 2/16 | 2/16 |
| **2** | 0 | 1/16 | 2/16 | 2/16 |
| **3** | 0 | 0 | 1/16 | 2/16 |
| **4** | 0 | 0 | 0 | 1/16 |

# Example – Uniform distribution on a unit disk

Suppose that a pair of random variables $\boxed{(X, Y)}$ is chosen uniformly from the set of real points $(x, y)$ such that $x^2 + y^2 \leq 1$

$\Omega_X = [-1, 1]$
$\Omega_Y = [-1, 1]$




This is a disk of radius $1$ which has area $\pi$

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$

$$\iint_{\substack{\text{circle} \\ \text{of radius } 1}} f_{XY}(x, y)\, dx\, dy = 1$$

$\leftarrow$ Area circle,

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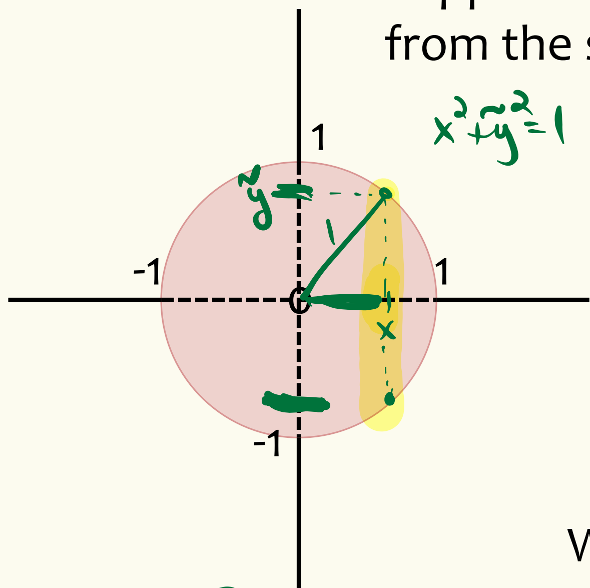
Are $X$ and $Y$ independent?
a. Yes
b. No



indep $\Leftrightarrow$ $f_{XY}(x,y) = f_X(x) f_Y(y)$ $\forall x, y$

# Example – Uniform distribution on a unit disk




Suppose that a pair of random variabes $(X, Y)$ is chosen uniformly from the set of real points $(x, y)$ such that $x^2 + y^2 \leq 1$

$x^2+\tilde{y}^2=1$ $\quad$ $\tilde{y}^2=1-x^2$

This is a disk of radius $1$ which has area $\pi$

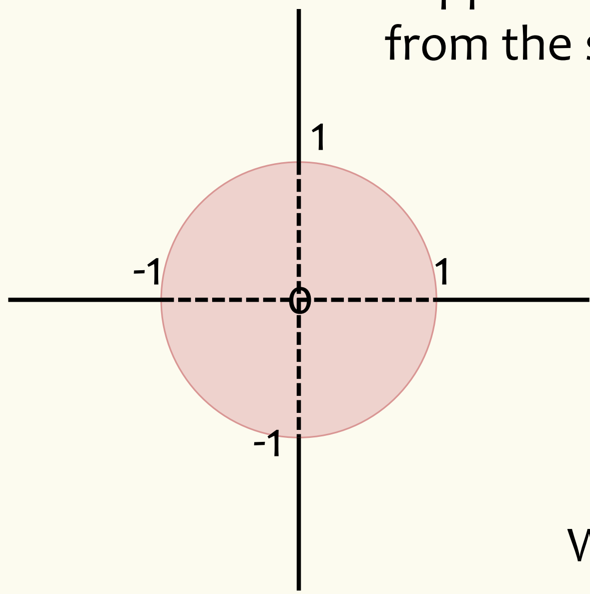
$$\boxed{f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}}$$

What is $f_X(x)$?

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$\boxed{f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy}$$

## Example – Uniform distribution on a unit disk




Suppose that a pair of random variabes $(X, Y)$ is chosen uniformly from the set of real points $(x, y)$ such that $x^2 + \mathrm{y}^2 \leq 1$

This is a disk of radius $1$ which has area $\pi$

$$f_{X,Y}(x, y) = \begin{cases} \dfrac{1}{\pi} & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

What is $f_X(x)$ ?

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} \mathrm{d}y$$

$$= 2\sqrt{1 - x^2}/\pi$$

## Joint Expectation

**Definition.** Let $X$ and $Y$ be discrete random variables and $p_{X,Y}(a,b)$ their joint PMF. The **expectation** of some function $g(x,y)$ with inputs $X$ and $Y$

$$\mathbb{E}[g(X,Y)] = \sum_{a \in \Omega_X} \sum_{b \in \Omega_Y} g(a,b) \cdot p_{X,Y}(a,b)$$

$E(X^2 Y^3)$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

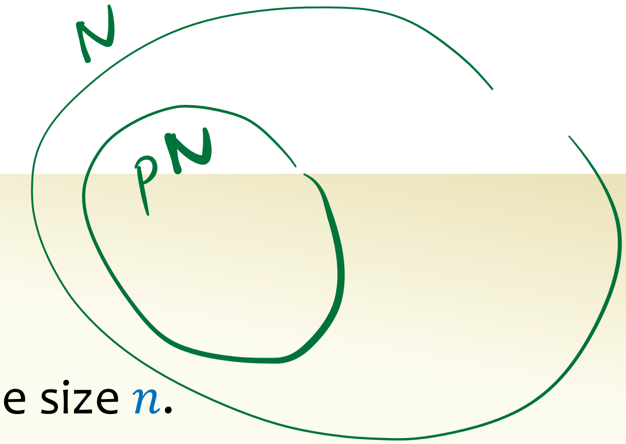
# Brain Break

# Agenda

- Joint Distributions
  - Cartesian Products
  - Joint PMFs and Joint Range
  - Marginal Distribution

- Polling

# Formalizing Polls




Population size $N$, true fraction of voting in favor $p$, sample size $n$.

**Problem:** We don't know $p$, want to estimate it

## Polling Procedure

for $i = 1, \ldots, n$:

1. Pick uniformly random person to call (prob: $1/N$)
2. Ask them how they will vote

$$X_i = \begin{cases} 1, & \text{voting in favor} \\ 0, & \text{otherwise} \end{cases}$$

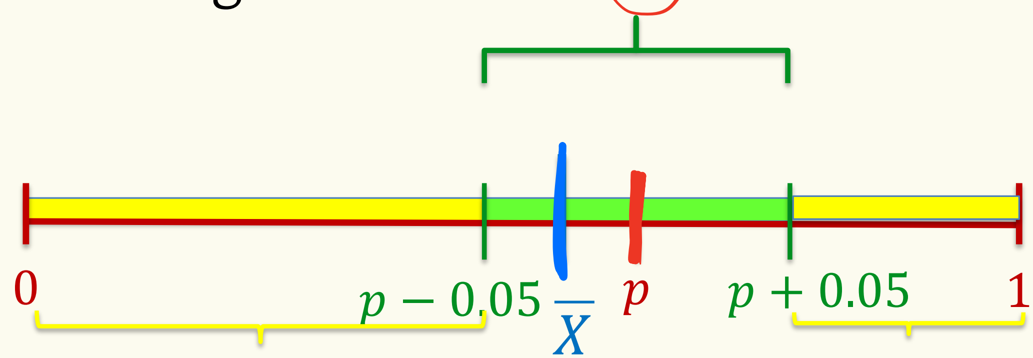
Report our estimate of $p$: $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

## Roadmap: Bounding Error

0.98

**Goal:** Find the value of $n$ such that 98% of the time, the estimate $\bar{X}$ is within 5% of the true $p$

Get good estimate if $\bar{X}$ lands in this region




Question: for what $n$ is $P(|\bar{X} - p| > 0.05) \le 0.02$

$( \quad \le 0.05) \ge 0.98$

## Central Limit Theorem

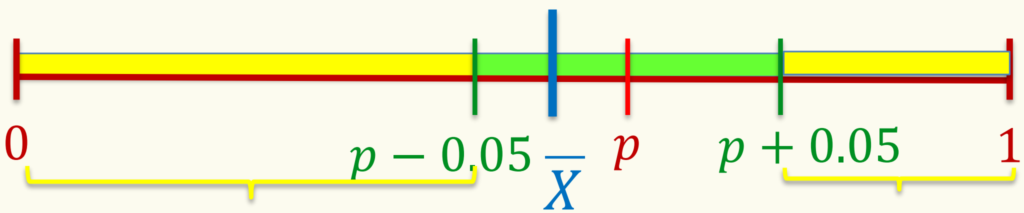
With i.i.d random variables $X_1, X_2, \ldots, X_n$ where $\mathbb{E}[X_i] = p$ and $\mathrm{Var}(X_i) = p(1-p)$

As $n \to \infty$,

$$\boxed{\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathcal{N}\left(p, \frac{p(1-p)}{n}\right)}$$

## Roadmap: Bounding Error

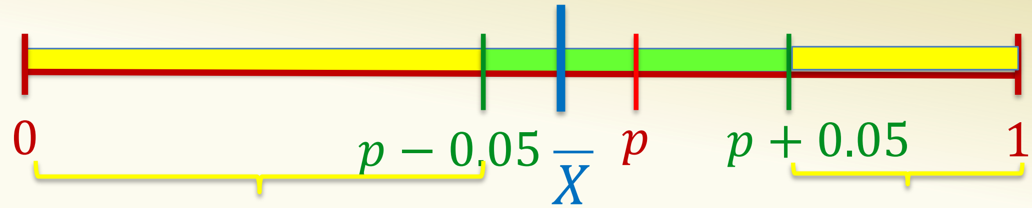
Question: for what $n$ is $P\left(\left|\overline{X} - p\right| > 0.05\right) \leq 0.02$




Crucial observation: the more samples we take, the more likely $\overline{X}$ is to be close to its expectation $p$ since as $n \to \infty$,

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathcal{N}\left(p, \frac{p(1-p)}{n}\right)$$

## Recap I




**Goal:** Find the value of $n$ such that 98% of the time, the estimate $\bar{X}$ is within 5% of the true $p$

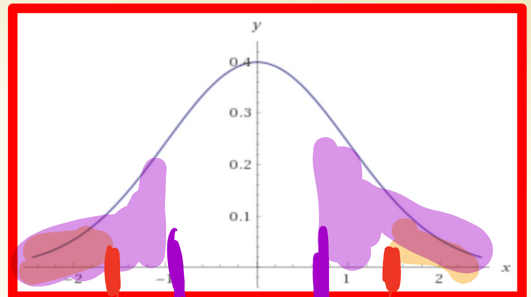
1. Define question. For what $n$ is $P(|\bar{X} - p| > 0.05) \leq 0.02$
2. Apply CLT: By CLT $\bar{X} \to \mathcal{N}(\mu, \sigma^2)$ where $\mu = p$ and $\sigma^2 = p(1-p)/n$
3. Convert to a standard normal. Specifically, define $Z = \frac{\bar{X}-\mu}{\sigma} = \frac{\bar{X}-p}{\sigma}$. Then, by the CLT $Z \to \mathcal{N}(0,1)$
4. Solve for $n$

# Recap II

1. For what $n$ is $P(|\overline{X} - p| > 0.05) \leq 0.02$

2. By CLT $\overline{X} \to \mathcal{N}(\mu, \sigma^2)$ where $\mu = p$ and $\boxed{\sigma^2 = p(1-p)/n}$

3. Define $Z = \frac{\overline{X}-\mu}{\sigma} = \frac{\overline{X}-p}{\sigma}$. Then, by the CLT $Z \to \mathcal{N}(0,1)$



$-a$ $\quad$ $a$

$-0.1\sqrt{n}$ $\quad$ $0.1\sqrt{n}$

$$P(|\overline{X} - p| > 0.05) = P\left(\frac{|\overline{X}-p|}{\sigma} > \frac{0.05}{\sigma}\right)$$

$|Z|$

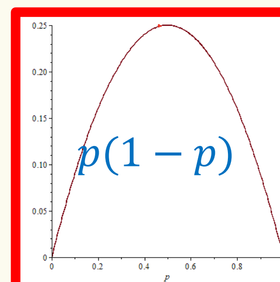
$Z \sim N(0,1)$

$\to a$

$$\frac{0.05}{\sqrt{\frac{p(1-p)}{n}}} = \boxed{\frac{0.05\sqrt{n}}{\sqrt{p(1-p)}}}$$

$$\geq 2 \cdot 0.05\sqrt{n} = 0.1\sqrt{n}$$

$\frac{1}{\sqrt{p(1-p)}}$ is always $\geq 2$

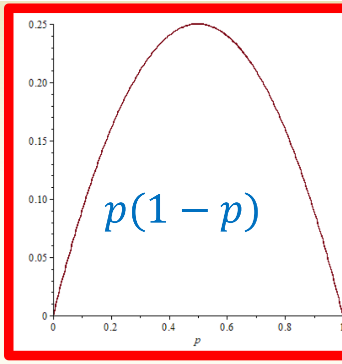



$p(1-p) \leq \frac{1}{4}$

$\sqrt{p(1-p)} \leq \frac{1}{2}$

$\sqrt{p(1-p)} \geq 2$

## Recap II




1. For what $n$ is $P\left(\left|\overline{X}-p\right|>0.05\right)\leq 0.02$
2. By CLT $\overline{X} \rightarrow \mathcal{N}(\mu, \sigma^2)$ where $\mu = p$ and $\sigma^2 = p(1-p)/n$
3. Define $Z = \frac{\overline{X}-\mu}{\sigma} = \frac{\overline{X}-p}{\sigma}$. Then, by the CLT $Z \rightarrow \mathcal{N}(0,1)$

$\boxed{\frac{\mathbf{1}}{\sqrt{\boldsymbol{p(1-p)}}} \text{ is always } \geq \mathbf{2}}$

$$P\left(\left|\overline{X}-p\right|>0.05\right) = P(|Z|\cdot\sigma > 0.05)$$

$$= P(|Z| > 0.05/\sigma) = P\left(|Z| > 0.05\frac{\sqrt{n}}{\sqrt{p(1-p)}}\right)$$

$$\leq P(|Z| > 0.1\sqrt{n})$$

Q: Why "$\leq$"?
A: This condition on $Z$
is easier to satisfy

## Recap III

1. Want $P\left(|\overline{X}-p|>0.05\right)\leq 0.02$

2. By CLT $\overline{X}\to\mathcal{N}(\mu,\sigma^2)$ where $\mu=p$ and $\sigma^2=p(1-p)/n$

3. Define $Z=\frac{\overline{X}-\mu}{\sigma}=\frac{\overline{X}-p}{\sigma}$. Then, by the CLT $Z\to\mathcal{N}(0,1)$

$p(1-p)$

1. Want $P\left(|\overline{X}-p|>0.05\right)\leq 0.02$

2. By CLT $\overline{X}\to\mathcal{N}(\mu,\sigma^2)$ where $\mu=p$ and $\sigma^2=p(1-p)/n$

3. Define $Z=\frac{\overline{X}-\mu}{\sigma}=\frac{\overline{X}-p}{\sigma}$. Then, by the CLT $Z\to\mathcal{N}(0,1)$

$\boxed{\frac{\mathbf{1}}{\sqrt{\boldsymbol{p(1-p)}}} \text{ is always } \geq \mathbf{2}}$

$$P\left(|\overline{X}-p|>0.05\right)=P(|Z|\cdot\sigma>0.05)$$

$$=P(|Z|>0.05/\sigma)=P\left(|Z|>0.05\frac{\sqrt{n}}{\sqrt{p(1-p)}}\right)$$

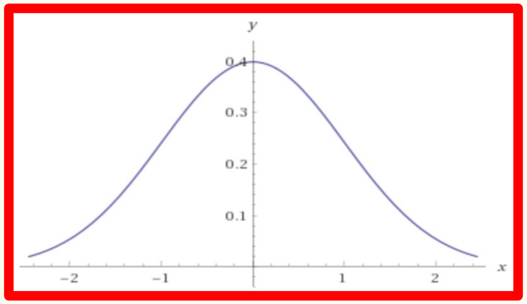
Want to choose $n$ so that this is at most 0.02

$$\leq P(|Z|>0.1\sqrt{n})$$

## Recap IV

Solve for $n$ such that $P(|Z| > 0.1\sqrt{n}) \leq 0.02$ *where* $Z \to \mathcal{N}(0, 1)$

- This assumes $n$ is large enough that $Z \sim \mathcal{N}(0, 1)$



## Recap V

We want $P(|Z| > 0.1\sqrt{n}) \leq 0.02$ *where* $Z \rightarrow \mathcal{N}(0, 1)$

- If we actually had $Z \sim \mathcal{N}(0, 1)$ then enough to show that $P(Z > 0.1\sqrt{n}) \leq 0.01$ since $\mathcal{N}(0, 1)$ is symmetric about $0$

- Use $P(Z > z) = 1 - \Phi(z)$ where $\Phi(z)$ is the CDF of the Standard Normal Distribution

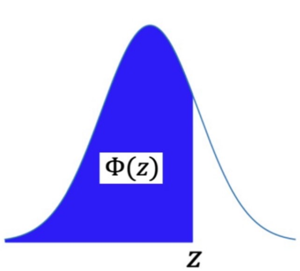
- Choose $n$ so that $0.1\sqrt{n} \geq z$ where $\Phi(z) \geq 0.99$

# Recap VI

## Table of $\Phi(z)$ CDF of Standard Normal Distribution

Choose $n$ so
$0.1\sqrt{n} \geq z$ where
$\Phi(z) \geq 0.99$

From table $z = 2.33$ works



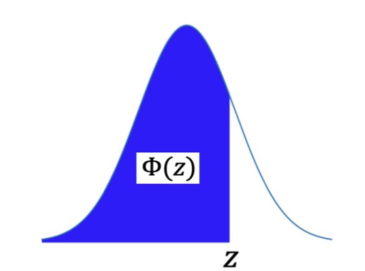

Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

| $z$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5 | 0.50399 | 0.50798 | 0.51197 | 0.51595 | 0.51994 | 0.52392 | 0.5279 | 0.53188 | 0.53586 |
| 0.1 | 0.53983 | 0.5438 | 0.54776 | 0.55172 | 0.55567 | 0.55962 | 0.56356 | 0.56749 | 0.57142 | 0.57535 |
| 0.2 | 0.57926 | 0.58317 | 0.58706 | 0.59095 | 0.59483 | 0.59871 | 0.60257 | 0.60642 | 0.61026 | 0.61409 |
| 0.3 | 0.61791 | 0.62172 | 0.62552 | 0.6293 | 0.63307 | 0.63683 | 0.64058 | 0.64431 | 0.64803 | 0.65173 |
| 0.4 | 0.65542 | 0.6591 | 0.66276 | 0.6664 | 0.67003 | 0.67364 | 0.67724 | 0.68082 | 0.68439 | 0.68793 |
| 0.5 | 0.69146 | 0.69497 | 0.69847 | 0.70194 | 0.7054 | 0.70884 | 0.71226 | 0.71566 | 0.71904 | 0.7224 |
| 0.6 | 0.72575 | 0.72907 | 0.73237 | 0.73565 | 0.73891 | 0.74215 | 0.74537 | 0.74857 | 0.75175 | 0.7549 |
| 0.7 | 0.75804 | 0.76115 | 0.76424 | 0.7673 | 0.77035 | 0.77337 | 0.77637 | 0.77935 | 0.7823 | 0.78524 |
| 0.8 | 0.78814 | 0.79103 | 0.79389 | 0.79673 | 0.79955 | 0.80234 | 0.80511 | 0.80785 | 0.81057 | 0.81327 |
| 0.9 | 0.81594 | 0.81859 | 0.82121 | 0.82381 | 0.82639 | 0.82894 | 0.83147 | 0.83398 | 0.83646 | 0.83891 |
| 1.0 | 0.84134 | 0.84375 | 0.84614 | 0.84849 | 0.85083 | 0.85314 | 0.85543 | 0.85769 | 0.85993 | 0.86214 |
| 1.1 | 0.86433 | 0.8665 | 0.86864 | 0.87076 | 0.87286 | 0.87493 | 0.87698 | 0.879 | 0.881 | 0.88298 |
| 1.2 | 0.88493 | 0.88686 | 0.88877 | 0.89065 | 0.89251 | 0.89435 | 0.89617 | 0.89796 | 0.89973 | 0.90147 |
| 1.3 | 0.9032 | 0.9049 | 0.90658 | 0.90824 | 0.90988 | 0.91149 | 0.91309 | 0.91466 | 0.91621 | 0.91774 |
| 1.4 | 0.91924 | 0.92073 | 0.9222 | 0.92364 | 0.92507 | 0.92647 | 0.92785 | 0.92922 | 0.93056 | 0.93189 |
| 1.5 | 0.93319 | 0.93448 | 0.93574 | 0.93699 | 0.93822 | 0.93943 | 0.94062 | 0.94179 | 0.94295 | 0.94408 |
| 1.6 | 0.9452 | 0.9463 | 0.94738 | 0.94845 | 0.9495 | 0.95053 | 0.95154 | 0.95254 | 0.95352 | 0.95449 |
| 1.7 | 0.95543 | 0.95637 | 0.95728 | 0.95818 | 0.95907 | 0.95994 | 0.9608 | 0.96164 | 0.96246 | 0.96327 |
| 1.8 | 0.96407 | 0.96485 | 0.96562 | 0.96638 | 0.96712 | 0.96784 | 0.96856 | 0.96926 | 0.96995 | 0.97062 |
| 1.9 | 0.97128 | 0.97193 | 0.97257 | 0.9732 | 0.97381 | 0.97441 | 0.975 | 0.97558 | 0.97615 | 0.9767 |
| 2.0 | 0.97725 | 0.97778 | 0.97831 | 0.97882 | 0.97932 | 0.97982 | 0.9803 | 0.98077 | 0.98124 | 0.98169 |
| 2.1 | 0.98214 | 0.98257 | 0.983 | 0.98341 | 0.98382 | 0.98422 | 0.98461 | 0.985 | 0.98537 | 0.98574 |
| 2.2 | 0.9861 | 0.98645 | 0.98679 | 0.98713 | 0.98745 | 0.98778 | 0.98809 | 0.9884 | 0.9887 | 0.98899 |
| 2.3 | 0.98928 | 0.98956 | 0.98983 | 0.9901 | 0.99036 | 0.99061 | 0.99086 | 0.99111 | 0.99134 | 0.99158 |
| 2.4 | 0.9918 | 0.99202 | 0.99224 | 0.99245 | 0.99266 | 0.99286 | 0.99305 | 0.99324 | 0.99343 | 0.99361 |
| 2.5 | 0.99379 | 0.99396 | 0.99413 | 0.9943 | 0.99446 | 0.99461 | 0.99477 | 0.99492 | 0.99506 | 0.9952 |
| 2.6 | 0.99534 | 0.99547 | 0.9956 | 0.99573 | 0.99585 | 0.99598 | 0.99609 | 0.99621 | 0.99632 | 0.99643 |
| 2.7 | 0.99653 | 0.99664 | 0.99674 | 0.99683 | 0.99693 | 0.99702 | 0.99711 | 0.9972 | 0.99728 | 0.99736 |
| 2.8 | 0.99744 | 0.99752 | 0.9976 | 0.99767 | 0.99774 | 0.99781 | 0.99788 | 0.99795 | 0.99801 | 0.99807 |
| 2.9 | 0.99813 | 0.99819 | 0.99825 | 0.99831 | 0.99836 | 0.99841 | 0.99846 | 0.99851 | 0.99856 | 0.99861 |
| 3.0 | 0.99865 | 0.99869 | 0.99874 | 0.99878 | 0.99882 | 0.99886 | 0.99889 | 0.99893 | 0.99896 | 0.999 |

# Recap VII

Choose $n$ so
$0.1\sqrt{n} \geq z$ where
$\Phi(z) \geq 0.99$

From table $z = 2.33$ works




- So we can choose $0.1\sqrt{n} \geq 2.33$ or $\sqrt{n} \geq 23.3$
- Then $n \geq 543 \geq (23.3)^2$ would be good enough … if we had $Z \sim \mathcal{N}(0, 1)$
- Since we only have $Z \to \mathcal{N}(0, 1)$ there is some loss due to approximation error (which can be dealt with).

## Summary: We found an approximate``confidence interval"

We are trying to estimate some parameter (e.g. $p$). We output an estimator $\overline{X}$ such that $P\left(\left|\overline{X} - p\right| > \epsilon\right) \leq \delta$ for some $(\epsilon, \delta)$.

- Often found using CLT, other approaches also important (especially when variance is unknown).
- We say that we are $(1 - \delta)$*100% confident that the result of our poll ($\overline{X}$) is an accurate estimate of $p$ to within $\epsilon$*100% percent.
- In our example, $(\epsilon = 0.05, \delta = 0.02)$.

## Idealized Polling

So far, we have been discussing "idealized polling". Real life is normally not so nice ☹

Assumed we can sample people uniformly at random, not really possible in practice

- Not everyone responds
- Response rates might differ in different groups
- Will people respond truthfully?

Makes polling in real life much more complex than this idealized model!