CSE 312

Foundations of Computing II

Lecture 3: More counting!

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

Recap (1)

Product Rule: In a sequential process, there are

- n_1 choices for the first step,
- n_2 choices for the second step (given the first choice), … , and
- n_m choices for the m^{th} step (given the previous choices),

then the total number of outcomes is $n_1 \times n_2 \times \cdots \times n_m$

Application. # of k-element sequences of distinct symbols (a.k.a. k-permutations) from n-element set is

$$P(n,k) = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

Recap (2)

Combination: If order does not matter, then count the number of ordered objects, and then divide by the number of orderings

Applications. The number of subsets of size k of a set of size n is

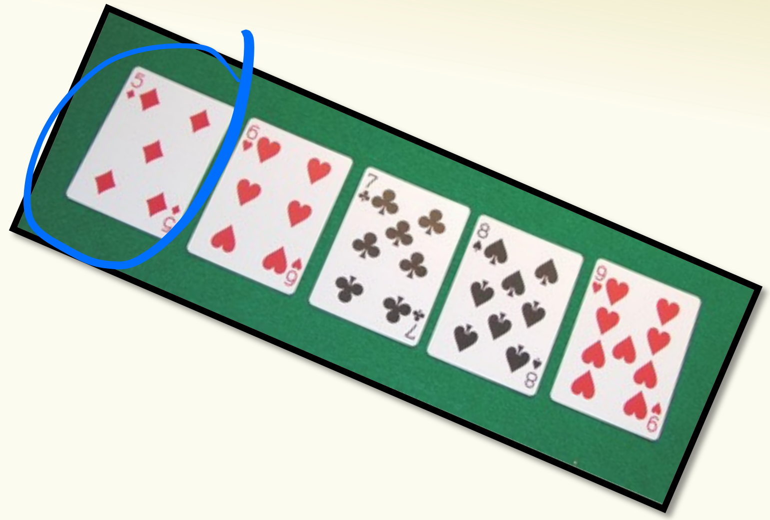
$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Binomial coefficient (verbalized as "n choose k")

Agenda

- More Examples + Sleuth's Criterion
- Stars and Bars
- Pigeonhole Principle

Quick Review of Cards



How many possible 5 card hands?

$\binom{52}{5}$

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

Counting Cards I

- 52 total cards
- 13 different ranks: A, 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

- A **straight** is five consecutive rank cards of any suit. How many possible straights?

Choose lowest rank 10

Choose suit 1st card 4

suit 2nd card 4

s

s

s 5th card. 4

$10 \cdot 4^5$

Counting Cards II

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

A **flush** is a five card hand all of the same suit.

How many possible flushes?

→ choose suit 4

→ pick 5 ranks. $\binom{13}{5}$

$4 \cdot \binom{13}{5}$

Counting Cards III

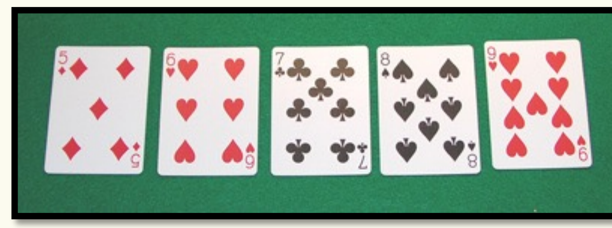
- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

A **flush** is five card hand all of the same suit.

How many possible flushes?

$$4 \cdot \binom{13}{5} = 5148$$

How many flushes are NOT straights?



flushes − # straight flushes

rank of least card then suit of all 5 cards

$$4\binom{13}{5} - 10 \cdot 4$$

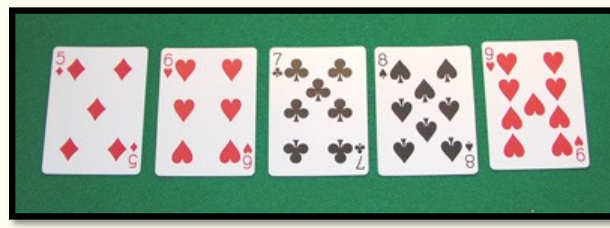
Counting Cards III

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

- A flush is five card hand all of the same suit.
  How many possible flushes?

$$4 \cdot \binom{13}{5} = 5148$$

- How many flushes are NOT straights?



= #flush - #flush and straight

$$\left(4 \cdot \binom{13}{5} = 5148\right) - 10 \cdot 4$$

Sleuth's Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence ➔ under counting

Many sequences ➔ over counting

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No sequence ➔ under counting   Many sequences ➔ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards.

$$\binom{4}{3} \cdot \binom{49}{2}$$

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Poll:

A. Correct

B. Overcount

C. Undercount

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No sequence ➔ under counting   Many sequences ➔ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

When in doubt, break up into disjoint sets you know how to count, and then use the sum rule.

Sleuth's Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence ➔ under counting Many sequences ➔ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

Use the sum rule

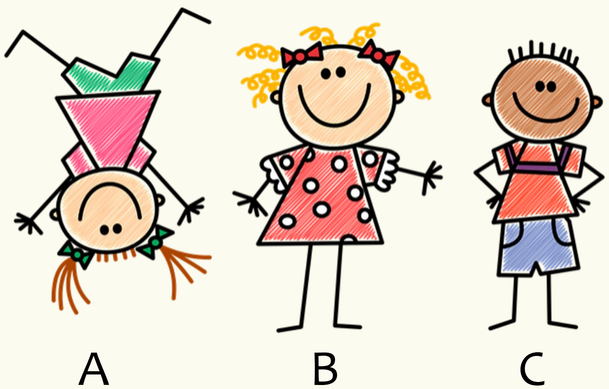
= # 5 card hand containing exactly 3 Aces $\binom{4}{3} \cdot \binom{48}{2}$

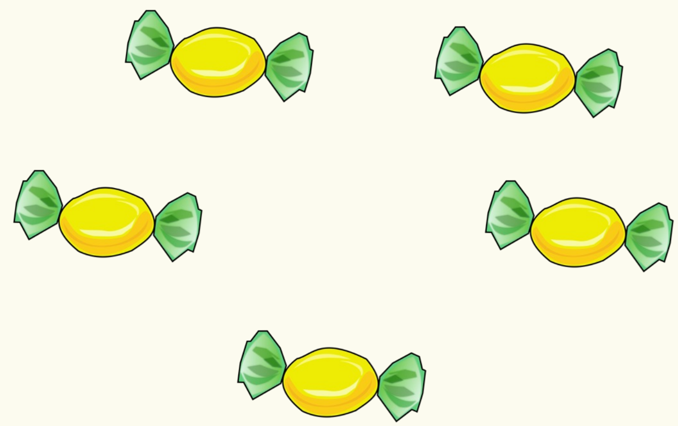
+ # 5 card hand containing exactly 4 Aces $\binom{48}{1}$

Agenda

- More Examples + Sleuth’s Criterion
- Stars and Bars
- Pigeonhole Principle

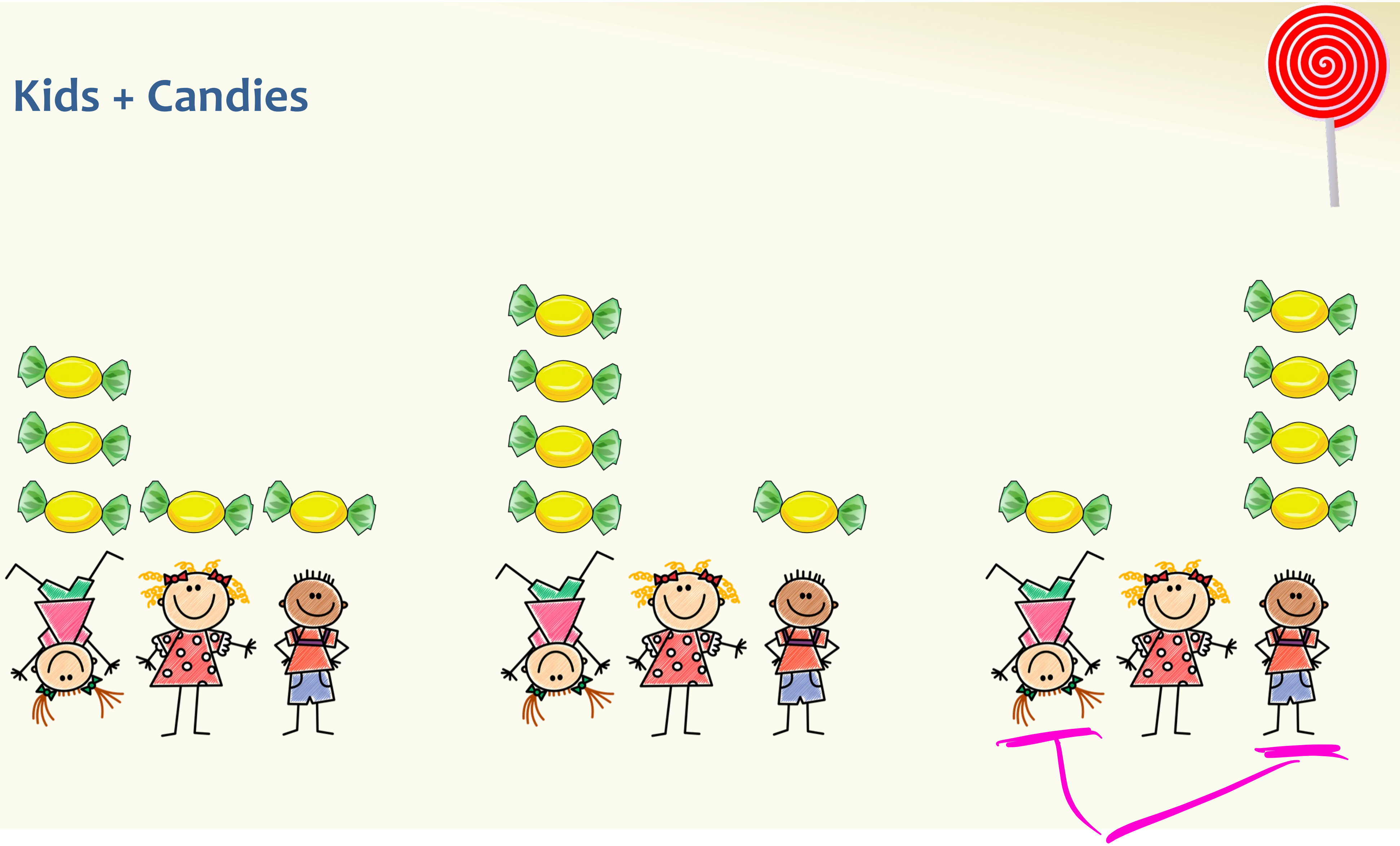
Example: Kids and Candies





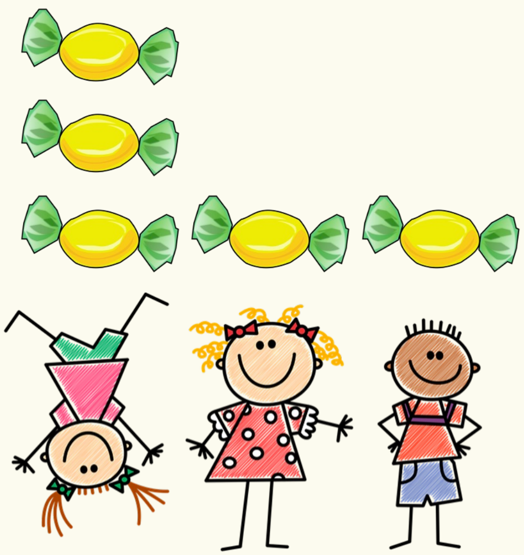
How many ways can we give five **indistinguishable** candies to these three kids?

Kids + Candies



Kids + Candies

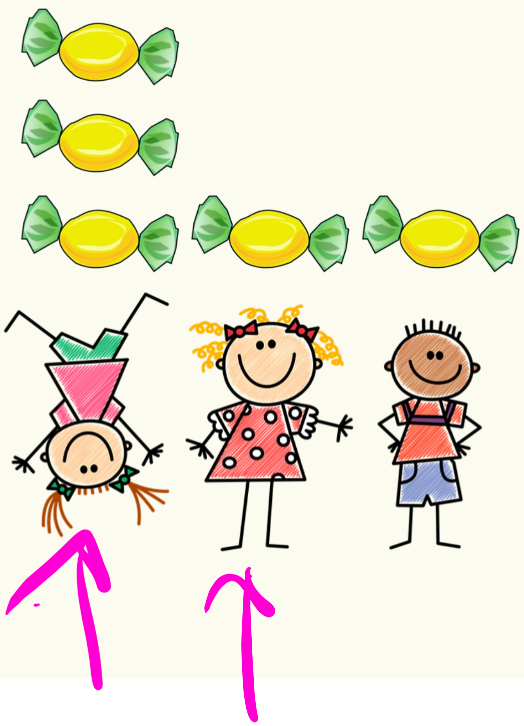
- Idea: count something different



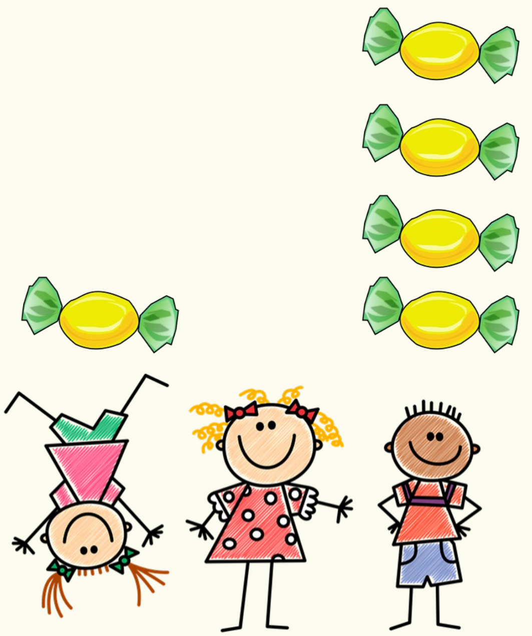
Kids + Candies

Idea: Count something equivalent

5 “stars” for candies, 2 “bars” for dividers.



Kids + Candies



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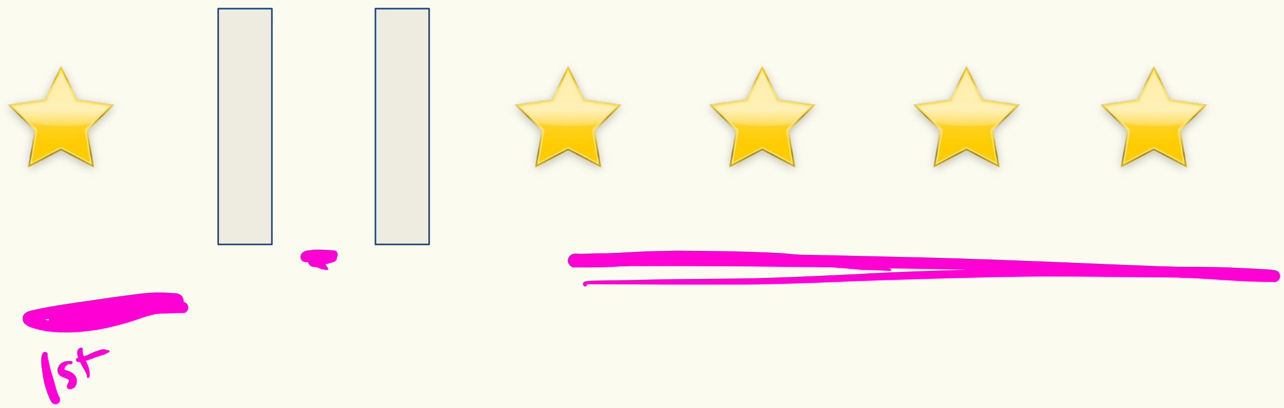
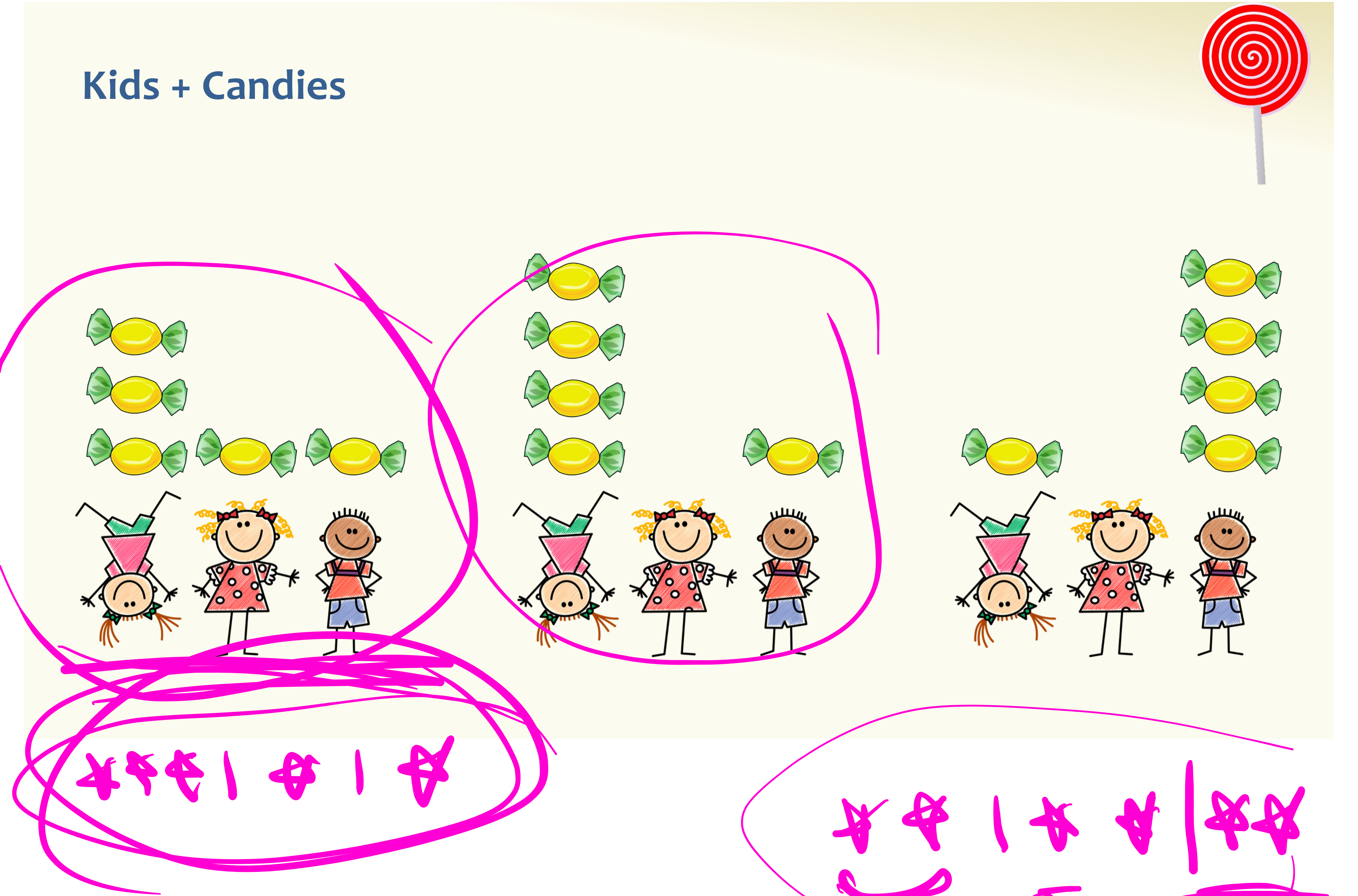
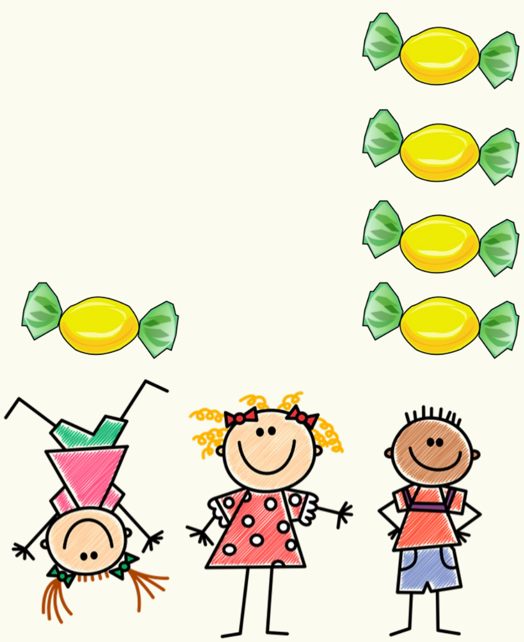


Kids + Candies



Kids + Candies

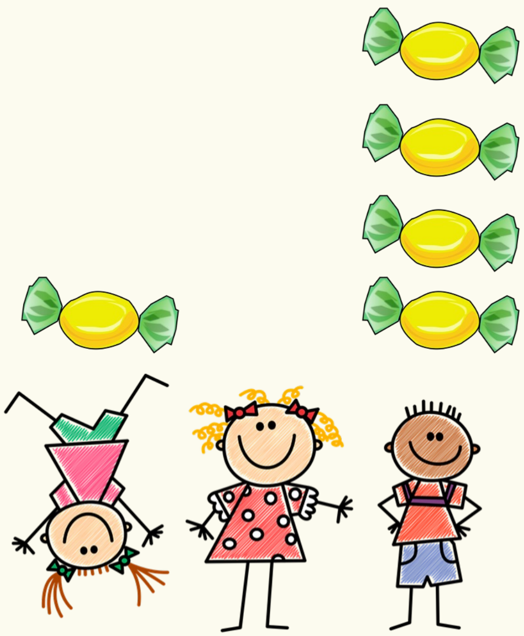


For each candy distribution, there is exactly one corresponding way to arrange the stars and bars.

Conversely, for each arrangement of stars and bars, there is exactly one candy distribution it represents.

$\binom{7}{2} = \binom{7}{5}$

Kids + Candies



Hence, the number of ways to distribute candies to the 3 kids is the number of arrangements of stars and bars.

This is

$$\binom{7}{2} = \binom{7}{5}$$

Stars and Bars / Divider method

The number of ways to distribute n indistinguishable balls into k distinguishable bins is

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

k = 6
n = 8

k-1 bars = dividers
n stars

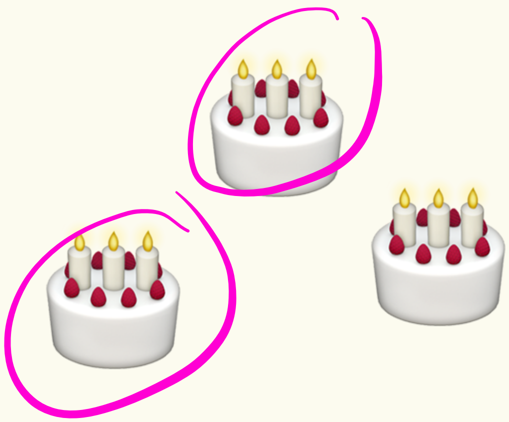
Agenda

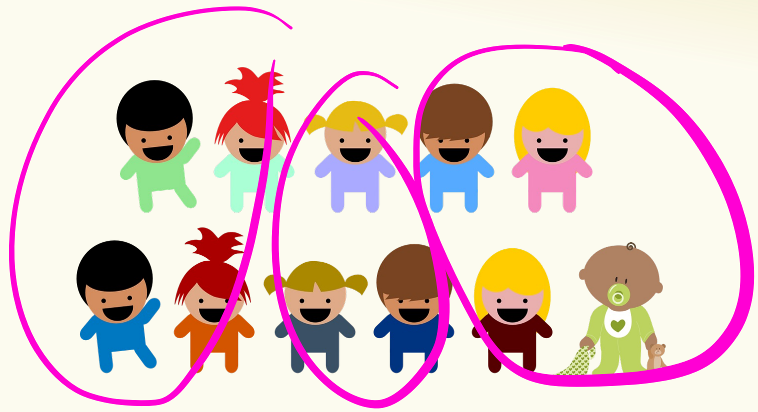
- More Examples + Sleuth's Criterion
- Stars and Bars
- Pigeonhole Principle

Pigeonhole Principle (PHP): Idea

10 pigeons, 9 pigeonholes

Pigeonhole Principle: Idea





If 11 children have to share 3 cakes, at least one cake must be shared by how many children?

$$\frac{11}{3} = 3\frac{2}{3}$$

Pigeonhole Principle – More generally

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\frac{n}{k}$ pigeons!

Proof. Assume there are $< \frac{n}{k}$ pigeons per hole.

Then, there are $< k\frac{n}{k} = n$ pigeons overall.

Contradiction!

$< k \cdot \frac{n}{k} = n$

Pigeonhole Principle – Better version

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons!

$\frac{n}{k}$ in some hole.

Pigeonhole Principle – Better version

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons!

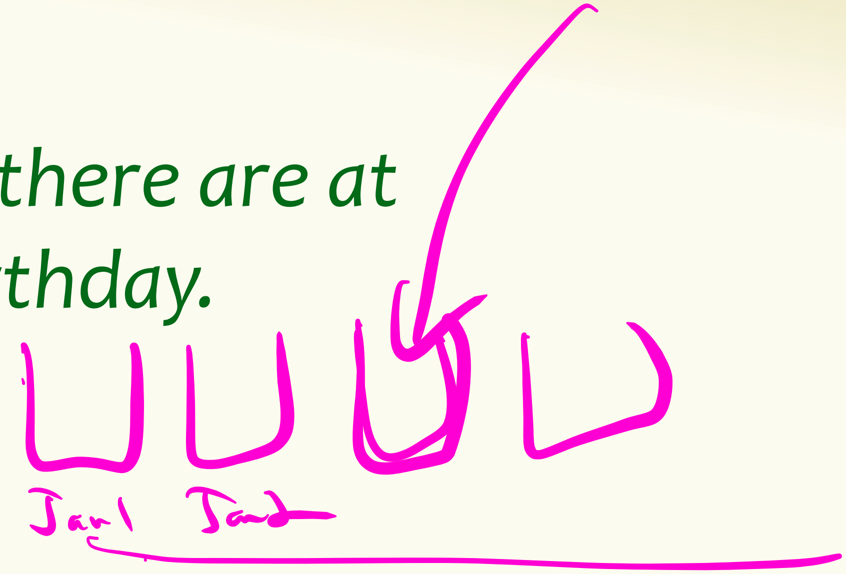
Reason. Can't have fractional number of pigeons

Syntax reminder:

- Ceiling: $\lceil x \rceil$ is x rounded up to the nearest integer (e.g., $\lceil 2.731 \rceil = 3$)
- Floor: $\lfloor x \rfloor$ is x rounded down to the nearest integer (e.g., $\lfloor 2.731 \rfloor = 2$)

Pigeonhole Principle – Example

In a room with 367 people, there are at least two with the same birthday.



Solution:

1. **367** pigeons = people
2. **365** holes = possible birthdays
3. Person goes into hole corresponding to own birthday
4. By PHP, there must be two people with the same birthday

$n = 367$

$k = 365$ pigeonholes $\lceil \frac{n}{k} \rceil$

Pigeonhole Principle: Strategy

To use the PHP to solve a problem, there are generally 4 steps:

1. Identify pigeons
2. Identify pigeonholes
3. Specify a rule for assigning pigeons to pigeonholes
4. Apply PHP

$2 \cdot 37 + 5 = i$

Pigeonhole Principle – Example (Surprising?)

In every set S of 100 integers, there are at least ***three*** *elements whose (pairwise) difference is a multiple of 37.*

When solving a PHP problem:

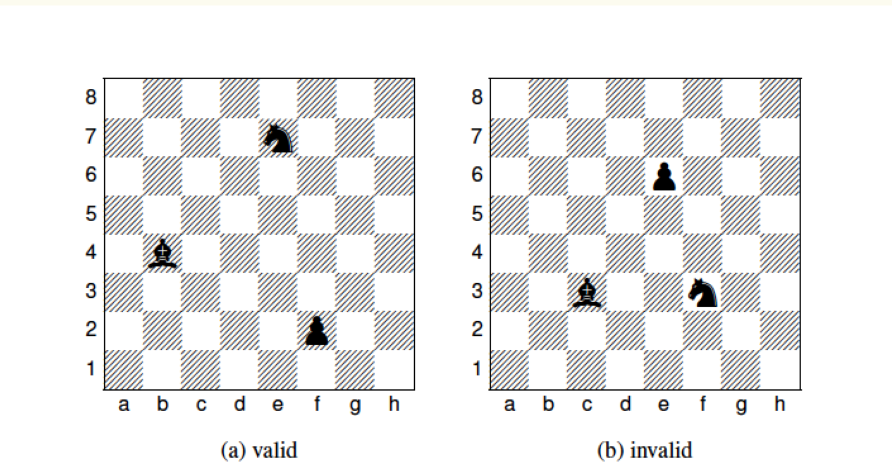
1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

Agenda

- Stars and Bars
- More Examples + Sleuth's Criterion
- Pigeonhole Principle
- Yet More Examples

8 by 8 chessboard

How many ways to place a pawn, a bishop, and a knight so that none are in the same row or column ?



Poll:

A. $\binom{64}{3}$

B. $\binom{8}{3} \cdot \binom{8}{3}$

C. $8^2 \cdot 7^2 \cdot 6^2$

D. I don't know.

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8 by 8 chessboard

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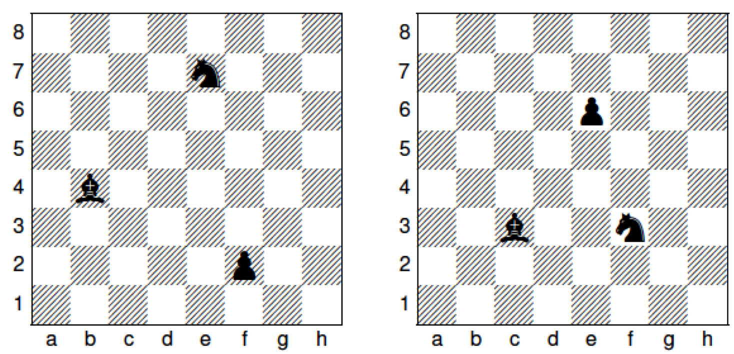


(a) valid (b) invalid

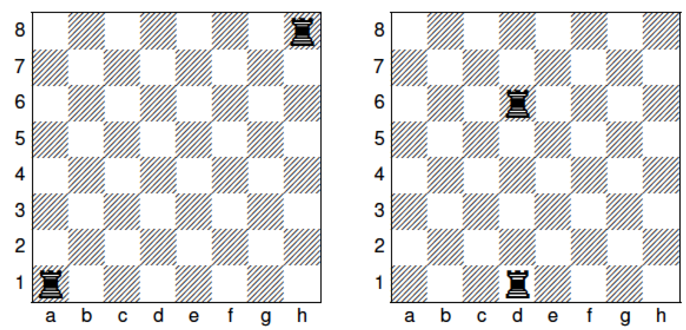
Sequential process:

1. Column for pawn
2. Row for pawn
3. Column for bishop
4. Row for bishop
5. Column for knight
6. Row for knight

$$(8 \cdot 7 \cdot 6)^2$$

Rooks on chessboard

How many ways to place two identical rooks on a chessboard so that they don't share a row or a column



(a) valid (b) invalid

Poll:

A. $8^2 \cdot 7^2$

B. $\binom{8}{2} \cdot \binom{8}{2}$

C. $\frac{8^2 \cdot 7^2}{2}$

D. I don't know.

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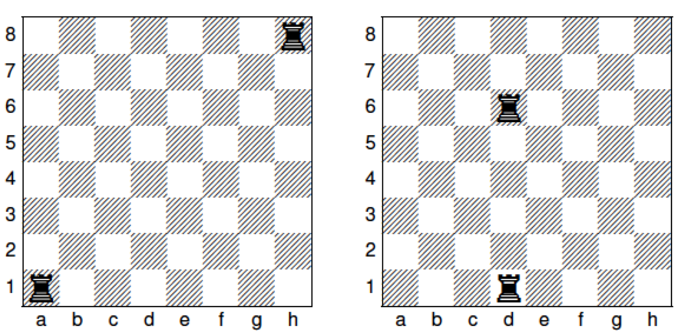
Rooks on chessboard

How many ways to place two identical rooks on a chessboard so that they don't share a row or a column



(a) valid (b) invalid

Pretend Rooks are different

1. Column for rook1
2. Row for rook1
3. Column for rook2
4. Row for rook2

$$(8 \cdot 7)^2$$

Remove the order between two rooks

$$(8 \cdot 7)^2/2$$

doughnuts

You go to top pot to buy a dozen donuts. Your choices are

Chocolate, Lemon-filled, Maple, Glazed, Plain

How many ways are there to choose a dozen doughnuts when doughnuts of the same type are indistinguishable?

doughnuts

You go to top pot to buy a dozen donuts. Your choices are

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How many ways are there to choose a dozen doughnuts when doughnuts of the same type are indistinguishable?

There is a bijection between sequences of length 16 with 4 bars and 12 stars and the doughnut choices.

doughnuts

You go to top pot to buy a dozen donuts. Your choices are

Chocolate, Lemon-filled, Maple, Glazed, plain

How many ways are there to choose a dozen doughnuts when you want at least 1 of each type?

doughnuts

You go to top pot to buy a dozen donuts. Your choices are

Chocolate, Lemon-filled, Maple, Glazed, plain

How many ways are there to choose a dozen doughnuts when you want at least 1 of each type?

Mental process:

1. Place one donut in each flavor bin
2. Choose the remaining 7 donuts without restriction

$$\binom{7+5-1}{5-1}$$

Challenge problem

Use the stars and bars approach to count the number of ways to choose k out of n distinct objects if repetition is allowed (but don’t care about order)

Tools and concepts

- Sum rule, Product rule
- Permutations, combinations
- Inclusion-exclusion
- Binomial Theorem
- Combinatorial proofs
- Pigeonhole principle
- Stars and bars