Problem Set 4

Due: Wednesday, January 31 by 11:59pm

Instructions

Solutions format and late policy. See PSet 1 for further details. The same requirements and policies still apply. Also follow the typesetting instructions from the prior PSets.

Collaboration policy. The written problems on this pset may be done with a **single partner**. In this case, only one person will submit the written part on Gradescope and add their partner as a collaborator.

Individuals and pairs are still encouraged to discuss problem-solving strategies with other classmates as well as the course staff, but each pair must write up their own solutions and, as stated above, submit a **single joint** homework. *However, you should make sure you are both involved in coming up with and writing up all the solutions.*

Solutions submission. You must submit your solution via Gradescope. In particular, submit under "PSet 4 [Written]" a **single** PDF file containing the solution to all tasks in the homework (for you and your partner). Each numbered task should be solved on its own page (or pages). Follow the prompt on Gradescope to link tasks to your pages. Do not write your names on the individual pages – Gradescope will handle that.

There is no programming on this problem set and you will not be turning in your solution to Task 10 should you choose to do it.

Task 1 – Heads vs Tails [10 pts]

Suppose that a coin with probability 0.6 of coming up heads is tossed independently 20 times. Let H be the number of times the coin comes up heads, let T be the number of times the coin comes up tails (i.e., $T = 20 - H$), and let $X = 3H - T$.

1. (5 points) What are the values of x such that $p_X(x) > 0$?
2. (5 points) What is $p_X(4)$, i.e., the probability that $X = 4$?

Task 2 – Laundry... [10 pts]

CSE 312 students sometimes delay laundry for a few days (to the chagrin of their roommates).
Suppose a busy 312 student must complete 3 problem sets before doing laundry. Each problem set requires 1 day with probability 2/3, and 2 days with probability 1/3. (The time it takes to complete different problem sets is independent.)
Let B be the number of days a busy student delays laundry. What is the probability mass function for B? (You don't need to simplify fractions.)

Task 3 – CDF to PMF [10 pts]

Suppose that X is a discrete random variable with cumulative distribution function (CDF)

$$F_X(x) = Pr(X \leqslant x) = \begin{cases} 0 & x < 0 \\ 1/3 & 0 \leqslant x < 1 \\ 7/18 & 1 \leqslant x < 2 \\ 5/9 & 2 \leqslant x < 3 \\ 2/3 & 3 \leqslant x < 4 \\ 1 & x \geqslant 4 \end{cases}$$

Find the probability mass function (pmf) for X. In other words, provide a formula for $p_X(x)$ that is correct for any x

Task 4 – Some Odd Dice [14 pts]

You are playing a game that uses a fair 8-sided die whose faces are numbered by the odd numbers, 1, 3, 5, . . . , 15. The value of a roll is the number showing on the top of the die when it comes to rest. Give all answers as simplified fractions.

1. (4 points) Let X be the value of one roll of the die. Compute $\mathbb{E}[X]$ and Var (X).

   NOTE: We will not cover variance until Friday, 1/26.

2. (4 points) Let Y be the sum of the values of 5 independent rolls of the die. Compute $\mathbb{E}[Y]$ and Var (Y). Use independence, and state precisely where in your computation you are using it.

3. (6 points) Let Z be the **average** of the values of n independent rolls of the die. Compute $\mathbb{E}[Z]$ and Var (Z). Use independence, and state precisely where in your computation you are using it.

Task 5 – Packet Failures [18 pts]

Consider three different models for sending n packets over the Internet:

1. (6 points) each packet takes a different path. Each path fails independently with probability p;

2. (6 points) all packets take the exact same path which fails with probability p. Thus, either all the packets get through or none get through;

3. (6 points) half the packets take one path, and half take the other (assume n is even), and each of the two paths fails independently with probability p.

Let X_i be the number of packets lost in case i, for i = 1, 2, 3. Write down the probability mass function, the expectation of X_i and the variance of X_i for $i = 1, 2, 3$.

Task 6 – A random word generator [10 pts]

Every minute, a random word generator spits out one word uniformly at random from the 3-word set {I , love, to}. The word spit out is independent of words spit out at other times. If we let the generator run for n minutes, what is the expected number of times that the phrase "I love to love" appears (assume $n \geqslant 4$)?

Task 7 – Jelly beans [15 pts]

You have 3 jars of jelly beans, one with n red jelly beans, one with n blue jelly beans and one with n green jelly beans. Each day when you come home, you eat one jelly bean from one of the jars, where the probability you eat a red one is 0.3, the probability you eat a blue one is 0.1 and the probability you eat a green one is 0.6. (The jar you choose from on each day is independent of the jar you choose from on any other day.) Let X be the total number of jelly beans that you've eaten at the end of the first day on which there is a jar with a single jelly bean left in it. Assume $n \geqslant 2$. (So if for example, $n = 5$, and you eat red, red, green, blue, red, green, blue, red and so on, then $X = 8$, but if you eat green, green, red, blue, blue, blue, green, red, red, green then $X = 10$.)

1. (3 points) What are the possible values X takes? (i.e., the set of numbers x such that $p_X(x) > 0$)?

2. (5 points) What is the probability that at the end of the first day on which there is a jar with a single jelly bean left in it you have eaten j red jelly beans, $n-1$ blue jelly beans and k green jelly beans (where $0 \leqslant j, k < n-1$)?

3. (7 points) What is $\mathbb{P}(X = n - 1 + m)$ where m is a nonnegative integer less than $n - 1$? (Simplify your answer using the binomial theorem.)

Task 8 – Tennis Tournament [10 pts]

Consider a tennis tournament in which each of the n people participating plays a match against every other person participating. (So there are $\binom{n}{2}$ matches.) Suppose that every match is equally likely to be won by either player, and the outcomes of different matches are independent. For a fixed permutation $\pi = (\pi_1, \pi_2, \pi_3, \pi_n)$ of the n people, we say that "π is ordered" if person π_1 wins their match against person π_2, and person π_2 wins their match against person π_3 and person π_3 wins their match against person π_4 and so on.
What is the expected number of "ordered" permutations (see definition above for "ordered" permutations)?

Task 9 – For Fun Problem - only if you're interested

This is a not-too-difficult challenge problem for those of you that would enjoy proving a cool probabilistic fact. You will not be turning anything in for this. It's just for fun.

You are shown two envelopes and told the following facts:

- Each envelope has some number of dollars in it, but you don't know how many.
- The amount in the first envelope is different from the amount in the second.
- Although you don't know exactly how much money is in each envelope, you are told that it is an integer number of dollars that is at least 1 and at most 100.
- You are told that you can pick an envelope, look inside, and then you will be given a one-time option to switch envelopes (without looking inside the new envelope). You will then be allowed to keep the money in envelope you end up with.

Your strategy is the following:

1. You pick an envelope uniformly at random.
2. You open it and count the amount of money inside. Say the result is x.
3. You then select an integer y between 1 and 100 uniformly at random.
4. If $y > x$, you switch envelopes, otherwise you stay with the envelope you picked in step (a)

Show that you have a better than 50-50 chance of taking home the envelope with the larger amount of money in it. More specifically, suppose the two envelopes have i and j dollars in them respectively, where $i < j$. Calculate the probability that you take home the envelope with the larger amount of money.