Problem Set 7

Due: Wednesday, February 28, by 11:59pm

Instructions

Solutions format and late policy. See PSet 1 for further details. The same requirements and policies still apply. Also follow the typesetting instructions from the prior PSets.

Collaboration policy. The written problems on this pset may be done with a **single partner**. In this case, only one person will submit the written part on Gradescope and add their partner as a collaborator.

Solutions submission. You must submit your solution via Gradescope under "PSet 7 [Written]". This will be a *single* PDF file containing the solution to all tasks in the homework. Each numbered task should be solved on its own page (or pages). Follow the prompt on Gradescope to link tasks to your pages. Do not write your name on the individual pages – Gradescope will handle that.

If you are working with a partner, there should be only one submission for both of you.

Task 1 – So Many Approximations [10 pts]

Let X be Binomial with parameters $n = 40$ and $p = 1/4$ and suppose that we are interested in computing $\mathbb{P}(5 \leqslant X \leqslant 7)$.

a) (3 points) Compute the answer exactly (correct to 4 decimal places).

b) (3 points) Approximate the answer using the Poisson approximation (correct to 4 decimal places).

c) (4 points) Approximate the answer using the Central Limit Theorem (that is, a normal approximation) with continuity correction (correct to 4 decimal places).

Task 2 – Distant Stars [15 pts]

An astronomer would like to measure the distance (in light years) from her observatory to a distant star. Each measurement is noisy, yielding only an estimate of the distance. Therefore, the astronomer plans to make a series of independent measurements and then use the average value of these measurements as her estimate of the actual distance. Suppose that each measurement has mean D (the true distance) and a variance of 4 light years. How many measurements does she need to make to be 95% confident that her estimate is accurate to within ± 0.5 light years?

Task 3 – Take me to a Higher Level **[25 pts]**

Shreya is playing League of Legends[1] for H hours, where H is a random variable, equally likely to be 1, 2 or 3. The level L that she gets to is random and depends on how long she plays for. We are told that

$$\mathbb{P}(L = \ell \mid H = h) = \frac{1}{h}, \qquad \text{for } \ell = 1, \ldots, h\,.$$

a) (5 points) Find the joint distribution of L and H.

b) (5 points) Find the marginal distribution of L.

c) (5 points) Find the conditional distribution of H given that $L = 1$ (that is, $\mathbb{P}(H = h \mid L = 1)$ for each possible h in 1,2,3). Use the definition of conditional probability and the results from previous parts.

d) (10 points) Suppose that we are told that Shreya got to level 1 or 2. Find the expected number of hours she played conditioned on this event, defined as follows:

$$\mathbb{E}\left[H \mid L = 1 \cup L = 2\right] = \sum_{h=1}^{3} h \cdot \mathbb{P}(H = h \mid L = 1 \cup L = 2)$$

Task 4 – Joint Densities **[15 pts]**

Suppose that X, Y are jointly continuous rv's with joint density

$$f_{X,Y}(x,y) = \begin{cases} 6x^2y & 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

(Observe that this is a probability density function since it is non-negative and we can use nested integrals to show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(xy)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_0^1 6x^2y\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \left(3x^2y^2\Big|_{y=0}^{y=1}\right)\mathrm{d}x = \int_0^1 3x^2\,\mathrm{d}x = x^3\Big|_{x=0}^{x=1} = 1.)$$

Your answers below should **not** be evaluated unless otherwise specified. Your answers should usually be in terms of integrals or nested double integrals.

a) (5 points) Write an expression using nested integrals that we can evaluate to find $\mathbb{P}(Y \geqslant X)$. Hint: draw the region of the joint density, and the desired region.

b) (5 points) Write an expression using a single integral that we can evaluate to find the marginal density $f_X(x)$. Be sure to specify the value of $f_X(x)$ for all $x \in \mathbb{R}$. Do the same for $f_Y(y)$.

c) (5 points) Are X and Y independent? Justify your answer. (You may need to evaluate an integral or two to do this.)

Task 5 – More joint densities **[16 pts]**

a) (4 points) Let Z be a uniformly random point in the unit radius disk centered at the origin of a plane in two dimensions. Let X (resp. Y) be the x-coordinate (respectively y-coordinate) of Z. Give a simple argument (of at most a few sentences) to show that X and Y are not independent.

[1] Anna wrote this problem never having played League of Legends so if it doesn't make sense in the context of the actual game, just pretend it does.

b) (12 points) Suppose that X and Y (not the same as in the previous part of this problem) have the following joint density:

$$f_{X,Y}(x,y) = \begin{cases} a & 0 < x \leqslant 0.5,\ 0 < y \leqslant 0.5 \\ b & 0 < x \leqslant 0.5,\ 0.5 < y < 1 \\ b & 0.5 < x \leqslant 1,\ 0 \leqslant y \leqslant 0.5 \\ a & 0.5 < x < 1,\ 0.5 < y < 1, \end{cases}$$

where a and b are constants.
- For what values of a and b are the marginal distributions of X and Y uniform on (0,1)?
- For what values of a and b are X and Y independent?
Justify your answers.